CH-01 COMPLEX NUMBERS

Let us consider the equation

$x^2+1=0$

$\Rightarrow x^2=-1$

$\Rightarrow x=\sqrt{-1}$

where $\sqrt{-1}=i$, the basic imaginary number.

Then $\sqrt{-4}=2i$

$\sqrt{-2}=\sqrt{2}i$ and so on.

→ Taking $i=\sqrt{-1}$, we observe that

$i^2=-1$

$i^3=-1\cdot i=-i$

$i^4=1$

Since $i^4=1$, $i=i^5=i^9=i^{13}=\dots=i^{4n+1}$

$i^2=i^6=i^{10}=i^{14}=\dots=i^{4n+2}$

$i^3=i^7=i^{11}=i^{14}=\dots=i^{4n+3}$

$i^4=i^8=i^{12}=i^{16}=\dots=i^{4n}$

Complex Numbers :-

The numbers of the form $a+ib$ where 'a' and 'b' are real numbers and $i=\sqrt{-1}$ are known as complex numbers.

→ In the complex number $z=a+ib$

The real numbers a & b respectively known as real and Imaginary parts of z.

→ so we can write $Re(z) = a$ and $Im(z) = b$.

i.e $C = \{z : z = a + ib,\ a, b \in \mathbb{R}\}$

Purely Real and Purely Imaginary Numbers :-

A complex number 'z' is said to be (i) purely real if $Im(z) = 0$

(ii) purely Imaginary if $Re(z) = 0$

Thus $2, -7, \sqrt{3}, \sqrt{5}$ etc are all purely real numbers.

→ while $2i$, $i\sqrt{3}$, $\frac{-1}{2}i$ etc are purely imaginary.

$$z = a + ib$$

Real part

Imaginary part

Conjugate of a complex Number :-

The conjugate of a complex number 'z', denoted by $\bar{z}$ & changing the sign of imaginary part.

i.e $z = a + ib$

$$\Rightarrow \bar{z} = \overline{a+ib} = a - ib$$

Ex:- $z = 2 + 3i$

$$\Rightarrow \bar{z} = \overline{2+3i} = 2 - 3i$$

Ex:- $z = 3 + 5i$

$$\Rightarrow \bar{z} = \overline{3+5i} = 3 - 5i$$

Ex:- $z = 6i$

$$\Rightarrow \bar{z} = \overline{6i} = -6i$$

Modulus of a Complex Number :-

If $z = x + iy$ be any complex Number, Then the modulus of z is written as $|z|$ is a real number $\sqrt{x^2+y^2}$

i.e $|z| = \sqrt{x^2+y^2}$

Ex:- If $z = 3+4i$. Find the modulus of z.

Soln :- $z = 3+4i$

$\Rightarrow |z| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

Ex:- Find the modulus and conjugate of $3-2i$.

Soln :- $z = 3-2i$

The modulus of $3-2i$ is

$|z| = \sqrt{3^2+(-2)^2} = \sqrt{9+4} = \sqrt{13}$

The conjugate of z is

$\bar{z} = \overline{3-2i} = 3+2i$. Ans

27.07.2020

① Addition of complex Number

Let $z_1 = x_1 + iy_1$

& $z_2 = x_2 + iy_2$

$z = \boxed{x} + i\boxed{y}$

$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$= (x_1 + x_2) + (iy_1 + iy_2)$

$= (x_1 + x_2) + i(y_1 + y_2)$

$\Rightarrow z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Properties of Addition

(i) Commutative

Let z_1, z_2 are two complex Nu.

$z_1 + z_2 = z_2 + z_1$

(ii) Associative

Let z_1, z_2, z_3 are three complex Nu.

$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

(iii) Existence of Additive identity:-

Let z be any complex Nu.

the $z + 0 = 0 + z = z$

(iv) Existence of Additive Inverse:-

Let z be any complex Nu.

then $z + (-z) = (-z) + z = 0$

Ex:- If $z_1 = 2 + 3i$ then find $z_1 + z_2$?
$z_2 = 3 + 5i$

Sol^n $z_1 = 2 + 3i$
$z_2 = 3 + 5i$

$z_1 + z_2 = (2 + 3i) + (3 + 5i)$

$= (2 + 3) + (3i + 5i)$

$= 5 + 8i$

$\Rightarrow z_1 + z_2 = 5 + 8i$

Ex:- Let $z_1 = 5 + 2i$. find $z_1 + z_2$?
$z_2 = 4 + 3i$

Sol^n:- $z_1 + z_2 = (5 + 2i) + (4 + 3i)$

$= (5 + 4) + (2i + 3i)$

$= 9 + 5i$

$\Rightarrow z_1 + z_2 = 9 + 5i$ Ans

Substraction of Two Complex Number

Let z_1 & z_2 are two complex Nu.

If $z_1 = x_1 + iy_1$
$z_2 = x_2 + iy_2$

$\Rightarrow z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$

$\Rightarrow z_1 - z_2 = (x_1 - x_2) + (i y_1 - i y_2)$

$$\boxed{\Rightarrow z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)}$$

Ex:- 1 If $z_1 = 4 + 2i$, $z_2 = 3 + 4i$.

Find $z_1 - z_2$?

Sol^n $z_1 = 4 + 2i$

$z_2 = 3 + 4i$

$z_1 - z_2 = (4 + 2i) - (3 + 4i)$

$= (4 - 3) + (2i - 4i) = 1 + i(2 - 4)$

$\Rightarrow z_1 - z_2 = 1 - 2i = 1 + (-2i)$

(2) If $z_1 = 5 - 2i$, $z_2 = 6 + 4i$. Find

$z_2 - z_1$?

Sol^n $z_1 = 5 - 2i$, $z_2 = 6 + 4i$

$z_2 - z_1 = (6 + 4i) - (5 - 2i)$

$= (6 - 5) + (4i - (-2i)) = 1 + 6i$

$\Rightarrow z_2 - z_1 = 1 + 6i$ Ans

Multiplication of Two complex Number

Let z_1, z_2 are two complex Number

then if $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$$

$$= x_1 \cdot x_2 + ix_1y_2 + iy_1x_2 + iy_1 \cdot iy_2.$$

$$= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$$

$$= x_1x_2 + ix_1y_2 + ix_2y_1 - y_1y_2 \quad (\because i^2 = -1)$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

$$\Rightarrow z_1 \cdot z_2 = \underbrace{(x_1x_2 - y_1y_2)}_{\text{Real}} + i\underbrace{(x_1y_2 + x_2y_1)}_{\text{Img.}}$$

Properties:-

(I) commutative

$z_1 \cdot z_2 = z_2 \cdot z_1$

(II) Distributive property

Let z_1, z_2 & z_3 are three complex Nums

$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

(III) Associative property:-

If z_1, z_2, z_3 are 3 complex Number

$z_1\cdot(z_2\cdot z_3) = (z_1\cdot z_2)\cdot z_3$

(iv) Multiplicative Indentive

If z be any complex number then $z\cdot 1 = 1\cdot z = z$.

(v) Multiplicative Inverse :-

If z be any complex number

the $z\cdot z^{-1} = z^{-1}\cdot z = 1$

$z^{-1} = \frac{1}{z}$, $z\cdot z^{-1} = z\cdot\frac{1}{z} = 1$

Ex :- If $z_1 = 2+3i$ & $z_2 = 3+4i$ .then express in $a+ib$ form . $(z_1\cdot z_2)$

Soln :- $z_1 = 2+3i$, $z_2 = 3+4i$

$z_1\cdot z_2 = (2+3i) + (3+4i)$

$= (2\times3 + 2\times4i) + (3i\times3 + 3i\times4i)$

$= 6 + 8i + (9i + 12i^2)$ $\because (i^2 = -1)$

$= 6 + 8i + 9i - 12$

$= 6 - 12 + 8i + 9i = -6 + 17i$

$\Rightarrow z_1 \cdot z_2 = -6 + 17i$ which is in form $a+ib$

(2) Express in $a+ib$ form if

$z_1 = 4+2i$ & $z_2 = 3+i$

Sol^n $z_1 = 4+2i$

$z_2 = 3+i$

$$z_1 \cdot z_2 = (4+2i)\cdot(3+i)$$

$$= (4\times 3 + 4\times i) + (2i\times 3 + 2i\times i)$$

$$= 12 + 4i + 6i + 2i^2 \quad (\because i^2 = -1)$$

$$= 12 + 4i + 6i - 2$$

$$= 12 - 2 + 4i + 6i$$

$$= 10 + 10i$$

$\Rightarrow z_1 \cdot z_2 = 10 + 10i$ in form $a+ib$

Division of complex Number :-

Let z_1 & z_2 are two complex Number

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

then $$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

Q:- If $1, \omega, \omega^2$ are the cube roots of unity then (i) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) = 9$

(ii) $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8) \ldots$ to $2n$ factors $= 2^{2n}$

(iii) $(1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5 = 32$

29.07.2020

Ex:- Express in $a+ib$ form when $\left(\frac{z_1}{z_2}\right)$

$z_1 = 2+4i$, $z_2 = 3+i$

$$\frac{z_1}{z_2} = \frac{2+4i}{3+i} = \left(\frac{P}{Q}\right)$$

$$= \frac{(2+4i)(3-i)}{(3+i)(3-i)}$$

$$= \frac{(2+4i)(3-i)}{3^2-(i)^2} = \frac{2\times3-2i+4i\times3-4i\times i}{9-i^2}$$

$$= \frac{6-2i+12i-4i^2}{9-i^2} \qquad [\because i^2 = -1]$$

$$= \frac{6+10i-4\times(-1)}{9-(-1)} = \frac{6+10i+4}{10}$$

$$= \frac{10+10i}{10} = \frac{\cancel{10}}{\cancel{10}} + \frac{\cancel{10}}{\cancel{10}}i$$

$$\Rightarrow \frac{z_1}{z_2} = 1+i \quad (a+ib)$$

Q:- Express in a+ib form if

$\Rightarrow$

$$\frac{z_1}{z_2} = \frac{3+5i}{2+i}$$

Sol^n:- $\dfrac{z_1}{z_2} = \dfrac{3+5i}{2+i}$

$$= \frac{(3+5i)(2-i)}{(2+i)(2-i)} = \frac{3\times2-3\times i+5i\times2-5i\times i}{2^2-i^2}$$

$$= \frac{6-3i+10i-5i^2}{4-(-1)}$$

$$= \frac{6+7i-5(-1)}{5} = \frac{6+7i+5}{5}$$

$$= \frac{11+7i}{5}$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{11+7i}{5} = \frac{11}{5} + \frac{7i}{5} \quad (a+ib)$$

Cube root of Unity :-

$$\boxed{\begin{array}{l} \text{Let } x^3-1=0 \\ \Rightarrow x^3 = 1 \end{array}}$$

so $x^3 = 1$

$\Rightarrow x^3 - 1 = 0$

$\Rightarrow x^3 - 1^3 = 0 \qquad a^3-b^3 = (a-b)(a^2+ab+b^2)$

$\Rightarrow (x-1)(x^2+x\cdot1+1) = 0$

14.08.20

$\Rightarrow (x-1)(x^2+x+1)=0$ $\quad \boxed{ax^2+bx+c=0}$ $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Either $x-1=0$ or $x^2+x+1=0$

$\boxed{\Rightarrow x=1}$ $\quad \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x^2+x+1=0$

So roots are $\frac{-1\pm\sqrt{1^2-4\times1\times1}}{2\times1}$

$= \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{-3}}{2}$ $\quad \because \sqrt{-3}=\sqrt{(-1)\times3} = \sqrt{3}\,i$

$= \frac{-1\pm\sqrt{3}i}{2}$

$= \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$

$\Rightarrow \omega = \frac{-1+\sqrt{3}i}{2}, \omega^2 = \frac{-1-\sqrt{3}i}{2}$

So the the roots of $\boxed{x = 1, \omega, \omega^2}$

The relation b/w $1, \omega, \omega^2$ is

$1+\omega+\omega^2=0$ and $1+\omega=-\omega^2$
$1+\omega^2=-\omega$, $\boxed{\omega+\omega^2=-1}$

$1\cdot\omega\cdot\omega^2=1$

$\Rightarrow \boxed{\omega^3=1}$ $\Rightarrow \boxed{\omega=\frac{1}{\omega^2}}$
$\Rightarrow \boxed{\omega^2=\frac{1}{\omega}}$

Pf :- $1+\omega+\omega^2=0$

L.H.S. $1+\omega+\omega^2$

$= 1+\frac{-1+\sqrt{3}i}{2}+\frac{-1-\sqrt{3}i}{2}$

$= 1+\frac{-1+\sqrt{3}i-1-\sqrt{3}i}{2}$

$= 1+\frac{-\cancel{2}}{\cancel{2}} = 1-1=0 =$ R.H.S.

(Proved)

Q:- If $1, \omega, \omega^2$ are the cube root of unity then show ~~tat~~ that

$(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)=9$

Proof:- L.H.S $=(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)$

$=(1-\omega)(1-\omega^2)(1-\omega^3\omega)(1-\omega^3\omega^2)$

$=(1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$ $[\because 1\cdot\omega\cdot\omega^2=1]$

$=(1-\omega)^2(1-\omega^2)^2$

$=[(1-\omega)(1-\omega^2)]^2$

$=[1\cdot1-1\cdot\omega^2-\omega\cdot1-\omega\cdot(-\omega^2)]^2$

$=[1-\omega^2-\omega+\omega^3]^2=[1-\omega^2-\omega+1]^2$

$=[2\underbrace{-\omega-\omega^2}_{1}]^2$ $\left[\begin{array}{l}\because 1+\omega+\omega^2=0\\ \Rightarrow 1=-\omega-\omega^2\end{array}\right]$

$=[2+1]^2=3^2=9 =$ R.H.S. (Proved)

Ex:- If $1, \omega, \omega^2$ are cube roots of unity then show that

$$(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)\cdots \text{to } 2n \text{ factors} = 2^{2n}.$$

$$L.H.S = (1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)\cdots \text{to } 2n \text{ factor}$$

$$= (1-\omega+\omega^2)(1-\omega^2+\omega^3.\omega)(1-\omega^3.\omega+\omega^6.\omega^2)\cdots 2n \text{ factor}$$

$$= (1-\omega+\omega^2)(1-\omega^2+1.\omega)(1-1.\omega+\omega^3\omega^3.\omega^2)\cdots 2n \text{ factor}$$

$$= (1-\omega+\omega^2)(1-\omega^2+\omega)(1-\omega+\omega^2)\cdots 2n \text{ factor}$$

$\because 1.\omega.\omega^2 = 1$

$\because 1+\omega+\omega^2 = 0$

$\Rightarrow 1+\omega^2 = -\omega$

$$= (1+\omega^2-\omega)(1+\omega-\omega^2)(1+\omega^2-\omega)(1+\omega-\omega^2)\cdots 2n \text{ factor}$$

$$= (-\omega-\omega)(-\omega^2-\omega^2)(-\omega-\omega)(-\omega^2-\omega^2)\cdots 2n \text{ factor}$$

$$= (-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2)\cdots 2n \text{ factor}$$

$$= 2^2\omega^3 \times 2^2\omega^3 \times \cdots \text{to } 2n \text{ factors}$$

$$= 2^2 \times 2^2 \times 2^2 \cdots \text{to } 2n \text{ factors} = 2^{2n} = R.H.S$$

(Proved)

31.07.2020

Q:- If $1, \omega, \omega^2$ are cube of unity then $(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5=32$

L.H.S $=(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5$

$=(1+\omega^2-\omega)^5+(1+\omega-\omega^2)^5$ $\left[\begin{array}{l} \because 1+\omega+\omega^2=0 \\ \& \ 1\cdot\omega\cdot\omega^2=1 \\ \Rightarrow \boxed{\omega^3=1} \end{array}\right]$ $\omega+\omega^2=-1$

$=(-\omega-\omega)^5+(-\omega^2-\omega^2)^5$

$=(-2\omega)^5+(-2\omega^2)^5$ $\quad 1+\omega+\omega^2=0 \Rightarrow \boxed{1+\omega^2=-\omega}$, $\boxed{1+\omega=-\omega^2}$

$=-32\omega^5-32\omega^{10}$

$=-32(\omega^5+\omega^{10})$

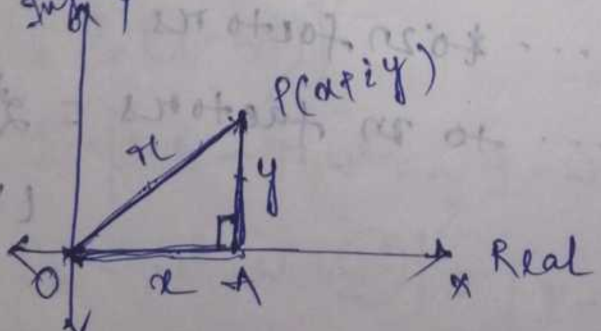
$=-32[\omega^3\cdot\omega^2+\omega^9\cdot\omega]$ $\quad \because \omega^9=(\omega^3)^3$

$=-32[1\cdot\omega^2+1\cdot\omega]$

$=-32[\omega+\omega^2]=-32\times(-1)=32=$ R.H.S (proved)

Polar form of complex Number:-

Let $z=x+iy$ be any complex Number



Here OAP be a right angle $\triangle$.

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$z = x + iy = r\cos\theta + ir\sin\theta$

$\Rightarrow \boxed{x + iy = r(\cos\theta + i\sin\theta)}$ → Polar form

$-\pi < \theta < \pi$

$\omega + \omega^2 = -1$

Ex:- Express $z = 1 + i$ into polar form.

$z = 1 + i$

we have $z = x + iy = r(\cos\theta + i\sin\theta)$

$\Rightarrow 1 + i = r(\cos\theta + i\sin\theta)$

$\Rightarrow \underset{(x+iy)}{1+i} = r\cos\theta + ir\sin\theta$

$x = r\cos\theta$	$\Rightarrow r\cos\theta = 1$ — (I)
$y = r\sin\theta$	$\Rightarrow r\sin\theta = 1$ — (II)

Again $\frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta} = \frac{1}{1}$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \tan^{-1} 1 = \frac{\pi}{4} \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

Again squaring and adding both (I) & (II)

$r^2\cos^2\theta + r^2\sin^2\theta = 1^2 + 1^2$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 2$

$\Rightarrow r^2 = 2 \Rightarrow \boxed{r = \sqrt{2}}$

(2)

05

$\Rightarrow z = x + iy$

$\Rightarrow z = 1 + i = r(\cos\theta + i\sin\theta)$

$$\boxed{\Rightarrow 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\pi/4\right)}$$

H.W

Ex:- Express $z = -1 + i\sqrt{3}$ in to polar form.

Ex:- Find the square root of $z = \sqrt{3+4i}$.

Soln:- $z = 3 + 4i$

Let $z = x + iy$ be any complex no. in which $x, y \in R$

Let $x + iy = \sqrt{3+4i}$

$\Rightarrow (x+iy)^2 = 3 + 4i$

$\Rightarrow x^2 + i^2y^2 + 2ixy = 3 + 4i$

$\Rightarrow (x^2 - y^2) + (2i)xy = 3 + 4i \quad (x+iy)$

$\Rightarrow \boxed{x^2 - y^2 = 3}$ and $2xy = 4$

If $z = x + iy$

Now squaring and adding both x & y

$(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2 = 3^2 + 4^2$

$\Rightarrow (x^2+y^2)^2 = 9+16 = 25$

$\Rightarrow \boxed{x^2+y^2 = \sqrt{25} = 5}$ — (2)

& $x^2 - y^2 = 3$ — (3)

From (2) & (3) we have

$x^2+y^2 = 5$
$x^2-y^2 = 3$

$2x^2 = 8$
$\Rightarrow x^2 = 4$
$\Rightarrow \boxed{x = \pm 2}$

$x=2$, $x^2+y^2=5$
$\Rightarrow 4+y^2 = 5$
$\Rightarrow y^2 = 5-4$
$\Rightarrow \boxed{y = \pm 1}$

So the square root of $3+4i$ is $2+i$.
Ans

Ex:- Find the square root of

$-5+12\sqrt{-1}$

Soln:- $z = x+iy = -5+12\sqrt{-1}$

$\Rightarrow x+iy = \sqrt{-5+12i}$

$\Rightarrow (x+iy)^2 = -5+12i$

$\Rightarrow x^2-y^2+i2xy = -5+12i$

where $x^2-y^2 = -5$ and $2xy = 12$

So $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= (5)^2 + 144$

$\Rightarrow (x^2+y^2)^2 = 25+144 = 169$

$\Rightarrow x^2+y^2 = \sqrt{169} = 13$

i.e $x^2-y^2 = -5$

$x^2+y^2 = 13$

$\Rightarrow 2x^2 = 8$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

Hence $y^2 = 9$

$\Rightarrow y = \pm 3$

so $\sqrt{-5+12\sqrt{-1}} = \pm(2 \pm 3i)$ Ans

Ex:- solve $z^7 = 1$

Soln $z^7 = \cos 0 + i \sin 0 = \cos 2k\pi + i \sin 2k\pi$

$z = \cos \frac{2k\pi}{7} + i \sin \frac{2k\pi}{7}$, $k = 0, 1, 2, 3, 4, 5, 6$

putting the value of k, we get

$k = 0$, $z = \cos 0 + i \sin 0$

$k = 1$, $z = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$

$k = 2$, $z = \cos \frac{4\pi}{7} + i \sin \frac{4\pi}{7}$

$k = 3$, $z = \cos \frac{6\pi}{7} + i \sin \frac{6\pi}{7}$

and so on.

→ De-Moivre's theorem says that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$n=0$, $(\cos\theta + i\sin\theta)^0 = 1$

$n=1$, $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

$n=2$, $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

General solution for De-Moivre's Thm

Thm is $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

→ If z be any complex number

then $z^n = 1$

Let $z = 1$ $(z = a+ib)$

$\Rightarrow z = 1 + 0.i \Rightarrow z = \cos\pi + i\sin\pi$

$\Rightarrow z^n = \cos 2k\pi + i\sin 2k\pi$

$\Rightarrow z = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$

$\Rightarrow \boxed{z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}}$ $[k = 0, 1, 2, \cdots n]$

→ If z be any complex number

then $z^n = -1$

Let $z = -1$ $(z = a+ib)$

$\Rightarrow z = -1 + 0.i$ $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

$\Rightarrow z = \cos\pi + i\sin\pi$

$\Rightarrow z^n = \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$

$\Rightarrow z^n = \cos\pi(2k+1) + i\sin\pi(2k+1)$

$\Rightarrow z = \left[\cos\pi(2k+1) + i\sin\pi(2k+1)\right]^{1/n}$

$\Rightarrow z = \left[\cos\pi\left(\frac{2k+1}{n}\right) + i\sin\pi\left(\frac{2k+1}{n}\right)\right]$

$k = 0, 1, 2, 3, \ldots n$

Ex:- Find the value for $z^n = 1$

$n = 0, 1, 2, 3, 4, 5, 6$

$n = 0$, $z^0 = 1$ $\quad x^{10} =$ $n = 0, 1, 2 \cdots 9$,

$n = 1$, $z^1 = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$

$= \cos 2k\pi + i\sin 2k\pi$

$= 1$

$n = 2$, $z^2 = \cos\frac{2k\pi}{2} + i\sin\frac{2k\pi}{2}$

$\Rightarrow z^2 = \cos k\pi + i\sin k\pi$

$= 1$

$n = 3$, $z^3 = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}$

Ex:- Multiply $\sqrt{-2} \times \sqrt{-3}$

Soln. $\sqrt{-2} \times \sqrt{-3}$ $\quad i^2 = -1 \Rightarrow i = \sqrt{-1}$

$\Rightarrow \sqrt{2}\sqrt{-1} \times \sqrt{3} \times \sqrt{-1}$

$= \sqrt{2}i \times \sqrt{3}i = \sqrt{6}\, i^2$

$= -\sqrt{6}$

Ex:- Multiply $(3\sqrt{-7} - 5\sqrt{-2})$ by $(3\sqrt{-2} + 5\sqrt{-2})$

Soln:- $(3\sqrt{-7} - 5\sqrt{-2}) \times (3\sqrt{-2} + 5\sqrt{-2})$ $\quad \left[\because i^2 = -1 \Rightarrow i = \sqrt{-1}\right]$

$= (3\sqrt{7}\cdot\sqrt{-1} - 5\sqrt{2}\cdot\sqrt{-1}) \times (3\sqrt{2}\times\sqrt{-1} + 5\sqrt{2}\times\sqrt{-1})$

$= (3\sqrt{7}i - 5\sqrt{2}i)(3\sqrt{2}i + 5\sqrt{2}i)$

$= 9\sqrt{14}\, i^2 + 15\sqrt{14}\, i^2 - 15\times 2\times i^2 - 25\times 2\cdot i^2$

$= -9\sqrt{14} - 15\sqrt{14} - 30 i^2 - 50 i^2$

$= -9\sqrt{14} - 15\sqrt{14} + 30 + 50$

$\Rightarrow 80 + 24\sqrt{14}$ Ans

Ex:- If $x+\frac{1}{x}=2\cos\theta$ then show that

$x^n+\frac{1}{x^n}=2\cos n\theta$

Soln :- $x+\frac{1}{x}=2\cos\theta$

$\Rightarrow \frac{x^2+1}{x}=2\cos\theta$

$\Rightarrow x^2+1=2x\cos\theta$

$\Rightarrow x^2+\cos^2\theta+\sin^2\theta=2x\cos\theta$ $[\because \cos^2\theta+\sin^2\theta=1]$

$\Rightarrow x^2+\cos^2\theta-2x\cos\theta=-\sin^2\theta$

$\Rightarrow (x-\cos\theta)^2=i^2\sin^2\theta$ $\because i^2=-1$

$\Rightarrow x-\cos\theta=\pm i\sin\theta$

$\Rightarrow \boxed{x=\cos\theta\pm i\sin\theta}$

If we choose $x=\cos\theta+i\sin\theta$ — (i) $\Rightarrow x^n$

$\Rightarrow \frac{1}{x}=\cos\theta-i\sin\theta$ — (ii)

~~Now adding (i) & (ii) we have~~

$x^n=(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$ — (i)

& $\left(\frac{1}{x}\right)^n=\frac{1}{x^n}=(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta$ — (ii)

Adding equ (i) & (ii) we have

$$x^n + \frac{1}{x^n} = \cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta}$$

$$\Rightarrow x^n + \frac{1}{x^n} = \cos n\theta + \cos n\theta$$

$$\Rightarrow \boxed{x^n + \frac{1}{x^n} = 2\cos n\theta} \quad \text{(proved)}$$

Ex! - If $x + \frac{1}{x} = 2\cos\theta$

then show that $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

Soln $x + \frac{1}{x} = 2\cos\theta$

$$\Rightarrow \frac{x^2+1}{x} = 2\cos\theta$$

$$\Rightarrow x^2 + 1 = 2x\cos\theta \quad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$\Rightarrow x^2 + \cos^2\theta + \sin^2\theta = 2x\cos\theta$$

$$\Rightarrow x^2 + \cos^2\theta - 2x\cos\theta = -\sin^2\theta$$

$$\Rightarrow (x - \cos\theta)^2 = i^2\sin^2\theta$$

$$\Rightarrow x - \cos\theta = \pm i\sin\theta$$

$$\Rightarrow x = \cos\theta \pm i\sin\theta$$

Now we can choose $x = \cos\theta + i\sin\theta$

$$\Rightarrow \frac{1}{x} = \cos\theta - i\sin\theta$$

$\Rightarrow x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ — (i)

& $\frac{1}{x^n} = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$ — (ii)

Now substracting eqn (ii) from (i)

$$x^n - \frac{1}{x^n} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$$

$$= \cancel{\cos n\theta} + i\sin n\theta - \cancel{\cos n\theta} + i\sin n\theta$$

$$\Rightarrow x^n - \frac{1}{x^n} = i\sin n\theta + i\sin n\theta$$

$$\boxed{\Rightarrow x^n - \frac{1}{x^n} = 2i\sin n\theta}$$ — (Proved)

(6) (27)

Ex:- 7.08.2020

(1) Find the value of $(1+i)^{100}$

Sol:- If $z = x+iy$ be any complex number then the polar form of

$z = r\cos\theta + ir\sin\theta$

$z = x+iy = 1+i$

$x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow x = 1, y = 1$

$r\cos\theta = 1$

$r\sin\theta = 1$

Now equating real part & Imaginary part we have

$r^2\cos^2\theta + r^2\sin^2\theta = 1^2+1^2$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 2$

$\Rightarrow r^2 = 2 \Rightarrow \boxed{r = \sqrt{2}}$

So $r\cos\theta = 1$

$\Rightarrow \cos\theta = \frac{1}{r}$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = 45^\circ = \frac{\pi}{4}$

$\boxed{\Rightarrow \theta = \frac{\pi}{4}}$

$r\sin\theta = 1$

$\Rightarrow \sin\theta = \frac{1}{r}$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = 45^\circ = \frac{\pi}{4}$

$1+i = r\cos\theta + ir\sin\theta$

$= \sqrt{2}\cos\frac{\pi}{4} + i\sqrt{2}\sin\frac{\pi}{4}$

$\Rightarrow 1+i = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\pi/4\right]$

$\Rightarrow (1+i)^{100} = (\sqrt{2})^{100}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^{100}$

$= (2)^{\frac{100}{2}}\left[\cos\frac{100\pi}{4} + i\sin\frac{100\pi}{4}\right]$

$\Rightarrow (1+i)^{100} = 2^{50}\left[\cos 25\pi + i\sin 25\pi\right]$

$\sin 180^\circ = 0$
$\sin\pi = 0$
$\Rightarrow \sin 2\pi = 0$
$3\pi = 0$

$= 2^{50}(-1+0)$

$\Rightarrow (1+i)^{100} = -2^{50}$ Ans

Ex:- Find the value of $(-i)^{4n+2}$

Soln:- $(-i)^{4n+2}$ $\quad \because i^4 = 1$

$= (-i)^{4n}\cdot(-i)^2$

$= 1\cdot -1 = -1$ Ans

Ex:- Multiply $\left(x - \frac{1+\sqrt{-3}}{2}\right)$ by $\left(x - \frac{1-\sqrt{3}}{2}\right)$

Soln:- $\left(x - \frac{1+\sqrt{-3}}{2}\right) \times \left(x - \frac{1-\sqrt{-3}}{2}\right)$

$\because w = \frac{1+\sqrt{3}i}{2}$

$i^2 = -1$ & $w^2 = \frac{1-\sqrt{3}i}{2}$

$\Rightarrow \sqrt{i^2} = \sqrt{-1}$

$= \left(x - \frac{1+\sqrt{3}i}{2}\right) \times \left(x - \frac{1-\sqrt{3}i}{2}\right)$

$= (x - w) \times (x - w^2)$

$= x^2 - x\omega^2 - \omega x + \omega^3$ $\quad \left[\begin{array}{l} \because 1+\omega+\omega^2 = 0 \Rightarrow \omega+\omega^2 = -1 \\ \& \ 1\cdot\omega\cdot\omega^2 = 1 \\ \Rightarrow \omega^3 = 1 \end{array}\right]$

$= x^2 - x(\omega+\omega^2) + 1$

$= x^2 - x(-1) + 1$

$= x^2 + x + 1$ Ans

Ex:- find the value of x & y if

$$x + y + i = 3 + (x-y)i$$

Sol^n:- $(x+y) + i = 3 + (x-y)i$

$\Rightarrow$ Now equating Real part with Real part & Imaginary part with imaginary part we have

$$x + y = 3 \quad -(1)$$

$$\& \quad x - y = 1 \quad -(11)$$

$$2x = 4$$

$$\boxed{\Rightarrow x = 2}$$

Now putting the value of 'x' in eqn (1)

we have $x + y = 3$

$\Rightarrow 2 + y = 3$

$\Rightarrow y = 3 - 2$

$$\boxed{\Rightarrow y = 1}$$

Ex:- If $x+\frac{1}{x}=2\cos\theta$. find the value of x.

Soln:- Given $x+\frac{1}{x}=2\cos\theta$

$\Rightarrow \frac{x^2+1}{x}=2\cos\theta$

$\Rightarrow x^2+1=2x\cos\theta$

$\Rightarrow x^2+\cos^2\theta+\sin^2\theta=2x\cos\theta$ $[\because \cos^2\theta+\sin^2\theta=1]$

$\Rightarrow x^2+\cos^2\theta-2x\cos\theta=-\sin^2\theta$

$\Rightarrow (x-\cos\theta)^2=i^2\sin^2\theta$ $[i^2=-1]$

$\Rightarrow x-\cos\theta=\pm i\sin\theta$

$\boxed{\Rightarrow x=\cos\theta\pm i\sin\theta}$ Ans

Q:- Simplify $\frac{1}{3-\sqrt{-2}}$

Soln:- $\frac{1}{3-\sqrt{-2}}$ $\left[\begin{array}{l} i^2=-1 \\ \Rightarrow i=\sqrt{-1} \end{array}\right]$

$=\frac{1}{3-\sqrt{2}\cdot\sqrt{-1}}=\frac{1}{3-\sqrt{2}i}$ $(3+\sqrt{2}i)$

$=\frac{1\,(3+\sqrt{2}i)}{(3-\sqrt{2}i)(3+\sqrt{2}i)}=\frac{3+\sqrt{2}i}{3^2-(\sqrt{2}i)^2}$ $(a-b)(a+b)=a^2-b^2$

$=\frac{3+\sqrt{2}i}{9-2i^2}=\frac{3+\sqrt{2}i}{9-2(-1)}=\frac{3+\sqrt{2}i}{11}$

$=\frac{3}{11}+\frac{\sqrt{2}i}{11}$ Ans

(9) (31)

Ex:- $(\cos\theta + i\sin\theta)^5 \cdot (\cos\theta - i\sin\theta)^3 = ?$

By De-Moivre's Thm

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$= (\cos 5\theta + i\sin 5\theta) \times (\cos 3\theta - i\sin 3\theta)$$

$$= \cos 5\theta \cdot \cos 3\theta - \cos 5\theta \cdot i\sin 3\theta + i\sin 5\theta \cdot \cos 3\theta - i^2 \sin 5\theta \cdot \sin 3\theta$$

$$= \cos 5\theta \cdot \cos 3\theta - i\sin 3\theta \cdot \cos 5\theta + i\sin 5\theta \cdot \cos 3\theta + \sin 5\theta \cdot \sin 3\theta \quad [\because i^2 = -1]$$

$$= \cos 5\theta \cdot \cos 3\theta + \sin 5\theta \cdot \sin 3\theta + i\sin 5\theta \cdot \cos\theta - i\sin 3\theta \cdot \cos 5\theta$$

$$= \cos(5\theta - 3\theta) + i\sin(5\theta - 3\theta)$$

$$= \cos 2\theta + i\sin 2\theta \quad \text{Ans}$$

CH-02 Matrices

08.08.2020

A matrix is rectangular array of numbers arranged in rows (Horizontal lines) and columns (vertical lines). If there are 'm' rows and 'n' of columns in a matrix, it is called an 'm' by 'n' matrix (or) the matrix of order $m \times n$.

Let
$$A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{array}{l} \to R_1 \\ \to R_2 \\ \vdots \\ \to R_m \end{array} \\ \begin{array}{ccccc} \downarrow & \downarrow & \downarrow & & \downarrow \\ C_1 & C_2 & C_3 & \cdots & C_n \end{array} \end{array}$$

It is denoted by (a_{ij}) or $[a_{ij}]$ where (a_{ij}) denotes the elements of ith row and jth column.

Ex:- $a_{ij} = a_{11}$ 1st row 1st column

Types of Matrix:-

① **Row Matrix:-** A matrix having a single row is called row matrix.

Ex:- $A = \begin{array}{l} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3} \to R_1 \\ \;\;\downarrow \;\; \downarrow \;\; \downarrow \\ \;\;C_1 \; C_2 \; C_3 \end{array}$, $A = \begin{bmatrix} a & b & c \end{bmatrix}_{1\times 3}$

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}_{1\times 4}$

(2) Column Matrix :- A Matrix having a single column is known as column Matrix.

It is denoted by $A = [a_{ij}] = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ \vdots \\ a_{1n} \end{bmatrix}$

Ex:- $A = \begin{array}{c} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \\ \downarrow \\ C_1 \end{array}$

(3) Zero Matrix :- A matrix in which all the elements are zero, is known as zero matrix.

Ex:- $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(4) Square Matrix:-

The matrix in which number of rows is equal to number of columns, is known as square Matrix.

Ex:- $a_{ij} = \begin{array}{c} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times3} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \\ \downarrow \quad \downarrow \quad \downarrow \\ C_1 \quad C_2 \quad C_3 \end{array}$

Ex:- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$, Ex:- $A = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 9 & 0 \\ 1 & 2 & 3 \end{bmatrix}_{3\times3}$

(5) Rectangular Matrix:-

A Matrix in which number of rows is not equal to no. of columns (or) viceversa is known as rectangular Matrix.

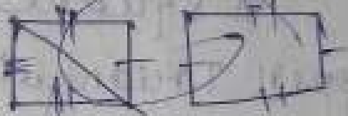
EX:- $A = \begin{array}{c} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \\ \begin{array}{cc} \downarrow & \downarrow \\ C_1 & C_2 \end{array} \end{array}$ (3×2) , $A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \end{array} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ C_1 & C_2 & C_3 \end{array} \end{array}$ (2×3)

$R \neq C$

(6) Diagonal Matrix:-

A square matrix is called a diagonal matrix if it's non-diagonal elements are zero.

EX:- $A = \begin{array}{c} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ C_1 & C_2 & C_3 \end{array} \end{array}$ 3×3 $3 = 3$



$A := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3} \simeq a_{ij} = \text{diag}[1, 2, 3]$

EX:- $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}_{2\times2}$

(7) Scalar Matrix:- A square Matrix is said to be scalar matrix, if all the diagonal elements are same.

EX:- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times3}$, $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}_{2\times2}$

(8) Unit Matrix (or) Identity Matrix :-

A square matrix is called unit or identity matrix if

(i) $a_{ij} = 0 \ \forall i \neq j$ and (ii) $a_{ij} = 1 \ \forall i = j$

i.e the leading diagonal elements are unity and rest others are zero.

Ex:- $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(9) Triangular Matrix:-

Upper Triangular:-

A square matrix (a_{ij}) is called upper Triangular Matrix if $a_{ij} = 0$, $i > j$

i.e the elements below the leading diagonals are zero.

$\therefore A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}_{3 \times 3}$

Lower Triangular Matrix

The elements above the leading diagonal are zero. i.e $a_{ij} = 0$, $i < j$

A:- $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}_{3 \times 3}$

⑩ Singular Matrix:-

A square matrix is said to be singular if it's determinant is zero, otherwise it is non-singular Matrix.

Ex:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = 1\times4 - 3\times2 = 4-6 = -2 \neq 0$

So it is not a singular Matrix.

⑪ Symmetric Matrix:-

A square Matrix is said to be symmetric if Transpose of A i.e $A^T = A$.

⑫ Skew - Symmetric Matrix:-

A square Matrix is said to be skew-symmetric if $A^T = -A$.

(13) Transpose of Matrix:- If A be any $m\times n$ Matrix, Transpose of A is a matrix of order $n\times m$ denoted by A^T or A'.

Ex:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{m\times n} \Rightarrow A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{n\times m}$

① $A:- \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3}$

$\Rightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times2}$.

Rank of Matrix :- A number 'r' is said to be the rank of Matrix of a non-zero $m \times n$ Matrix, if

(i) there is atleast one $r(r \times r)$ sub-Matrix of 'A' whose determinant is not equal to zero.

(ii) The determinant of every $(r+1)$ rowed square sub-Matrix in 'A' is zero.

→ The rank 'r' of the Matrix 'A' is denoted $r(A)$.

→ The rank of a non-singular square Matrix of order 'n' is 'n' and that of a singular matrix of order 'n' is less than 'n'.

→ If all the elements of 'A' are zero then $r(A) = 0$. i.e the rank of a matrix is assumed to be zero.

Ex:- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{matrix} \to R_1 \\ \to R_2 \end{matrix}$ (2×2) If $|A| \neq 0$

$C_1 \quad C_2 \quad m \times n$ the rank $(A) = 2$

order = 2

Ex:- If $A = \begin{array}{c|l} & \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \\ & \quad \downarrow c_1 \quad \downarrow c_2 \quad \downarrow c_3 \end{array}$ Then order of (A) = 3

(3×3)

$|A| \neq 0$ the order of matrix is known as the rank of matrix.

But $|A| = 0$, at that time for calculating the rank, we choose the sub-matrix of the given matrix.

→ <u>Sub-Matrix</u> :- If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times 3}$

submatrix are, $\begin{bmatrix} a & b \\ d & e \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} b & c \\ e & f \end{bmatrix}$

$\begin{bmatrix} d & e \\ g & h \end{bmatrix}$ and $\begin{bmatrix} a & f \\ h & i \end{bmatrix}$?..

Ex:- (1) Find the rank of matrix if

$A = \begin{matrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{matrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}_{3\times 3}$

so $r(A) \leq (3, 3)$

$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 1(3\times 5 - 4\times 4) - 2(2\times 5 - 3\times 4) + 3(2\times 4 - 3\times 3)$

$= 15 - 16 - 2(10 - 12) + 3(8 - 9)$

(7) (2)

$= 1 + 4 - 3 = 5 - 3 = 2 \neq 0$

so the rank of Matrix is '3'.

Ex:- (2) find the rank of Matrix if

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}_{3\times3}.$$

Soln :- $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}_{3\times3} \neq 0$

so the rank(A) $\leq (3,3)$

$\Rightarrow |A| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 0(0-2) - 0(4-0) + 1(0-2)$

$= 0 - 0 - 2 = -2 \neq 0$

so the rank of (A) = 3

Ex:-3 Find the rank of Matrix if

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 2 & 6 \end{bmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array} 3\times4 \\ \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \end{array}$$

Soln :- $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 2 & 6 \end{bmatrix}_{3\times4}$

$A_1 = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix}$

(B)(8)

$$A_4 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 4 & 2 & 6 \end{bmatrix}_{3\times 3}$$

$$\text{so } |A_4| = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 2 & 4 & 2 \end{vmatrix} = 1(2\times 2 - 4\times 1) - 2(2\times 1 - 1\times 2) + 0(1\times 4 - 2\times 2)$$

$$= 0$$

so $r(A) \leq (3,3)$

$|A_1| = |A_2| = |A_3| = |A_4| = 0$

Now we choose the sub matrix of the given matrix,

Let the sub-matrix of A_1 matrix is

$$\begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} = 2\times 1 - 2\times 0 = 2 \neq 0$$

so the rank of matrix $r(A) = \underline{2}$.

Ex:- Find the rank of matrix of

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}_{3\times 4}, \text{ so rank}(A) \leq (3,4)$$

$$A_1 = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 1 & 3 & 4 \end{bmatrix}_{3\times 3}, A_2 = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}_{3\times 3}, A_3 = \begin{bmatrix} 1 & 4 & 3 \\ 3 & 12 & 3 \\ 1 & 4 & 1 \end{bmatrix}_{3\times 3}$$

$$A_4 = \begin{bmatrix} 3 & 4 & 3 \\ 9 & 12 & 3 \\ 3 & 4 & 1 \end{bmatrix}_{3\times 3}$$

so $|A_4| = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 1 & 3 & 4 \end{bmatrix}_{3\times3} = 1(9\times4 - 12\times3) - 3(3\times4 - 1\times12) + 4(3\times3 - 9\times1)$

$\Rightarrow |A_4| = 0$

Again $|A_2| = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$

$= 1(9 - 3\times3) - 3(3\times1 - 1\times3) + 3(3\times3 - 1\times9)$

$= 0$

So $|A_1| = |A_2| = |A_3| = |A_4| = 0$

So we choose the submatrix of Given Matrix.

Let submatrix of A_2 is $\begin{bmatrix} 3 & 3 \\ 9 & 3 \end{bmatrix}_{2\times2}$

$\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} = 3\times3 - 9\times3 = 9 - 27$

$= -18 \neq 0$

So rank of A is 2.

Elementary Transformation :- 12.08.2020

The matrices obtained from a unit matrix 'I' after one (or) more elementary row or column operations are called elementary transformation.

Ex:- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times3}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Canonical Matrix :-

The matrix obtained by applying a series of elementary row/column operations such that there are some non-zero rows in the top and remaining rows consists of all zeros is called canonical matrix.

Ex:- $\begin{array}{c} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times3} \begin{array}{l} \rightarrow R \\ \rightarrow R \\ \\ \end{array} \\ \downarrow \\ C_1 \end{array} \rightarrow \begin{bmatrix} a & b & c \\ 0 & 0 & e \\ 0 & 0 & 0 \end{bmatrix}$

Ex:- $A = \begin{bmatrix} a & b & c \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

Methods for elementary transformation:-

(i) Interchange of any two rows or columns.

They are denoted by $R_i \leftrightarrow R_j$ (or)

$c_i \leftrightarrow c_j$

ith → row

jth → colum

Ex:- $A = \begin{array}{c} \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \end{array} \\ \begin{array}{cc} \downarrow & \downarrow \\ c_1 & c_2 \end{array} \end{array} \longleftrightarrow \begin{bmatrix} c & d \\ a & b \end{bmatrix}_{2\times2}$

$\xleftrightarrow{c_1 \leftrightarrow c_2} \begin{bmatrix} b & a \\ d & c \end{bmatrix}_{2\times2}$

Ex:- $A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ c_1 & c_2 & c_3 \end{array} \end{array}$

$\xleftrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{c_1 \leftrightarrow c_3} \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{bmatrix}$

(ii) Multiplication of ith row or jth column by non-zero number 'k', which denoted by $R_i \rightarrow kR_i$ (or) $c_j \rightarrow kc_j$

Ex:- $A = \begin{array}{c} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \begin{array}{l} \to R_1 \\ \to R_2 \end{array} \\ \begin{array}{cc} \downarrow & \downarrow \\ c_1 & c_2 \end{array} \end{array}$, Let k be any non-zero element

If k is multiplied with R_1, we have $R_1 \times k$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_1 \times k} \begin{bmatrix} 1k & 2k \\ 3 & 4 \end{bmatrix}$$

$$\xrightarrow{R_2 \times k} \begin{bmatrix} 1 & 2 \\ 3k & 4k \end{bmatrix}$$

Again $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{k \times c_1} \begin{bmatrix} 1k & 2 \\ 3k & 4 \end{bmatrix}$

$$\xrightarrow{k \times c_2} \begin{bmatrix} 1 & 2k \\ 3 & 4k \end{bmatrix}$$

Ex:- $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}_{2\times 2}$ & $k = 3$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \xrightarrow{R_1 \times 3} \begin{bmatrix} 2\times 3 & 3\times 3 \\ 4 & 5 \end{bmatrix} \to \begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix}$$

(iii) Adding a row or column by multiplying a non-zero element 'k' to a row (or) column. which is denoted by

$R_i \to R_i + k R_j$. and $C_i \to C_i + k C_j$

Ex:- $A = \begin{array}{c} \\ \end{array}\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{array}{c} R_1 \\ R_2 \end{array} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 3 \\ 2+2\times1 & 5+2\times3 \end{bmatrix}$

$\quad\quad C_1 \quad C_2$

$R_2 \to R_2 + 2R_1$

$\to \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \xrightarrow{C_2 \to C_2 + 3C_1} \begin{bmatrix} 1 & 3+3\times1 \\ 2 & 5+3\times2 \end{bmatrix}$

$\quad C_1 \quad C_2$

$\to \begin{bmatrix} 1 & 6 \\ 2 & 11 \end{bmatrix}$

Q:- Find the rank of Matrix if $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$

Elementary transformation

operation on rows/columns

canonical form Matrix

Ex:- $\begin{bmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix}_{3\times3} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$

Ex:- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Find the rank of matrix by elementary transformation.

Ex:- find the rank of Matrix, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}_{3\times3}$ by Elementary transformation.

Sol^n $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}_{3\times3} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$ $\begin{bmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix}$ canonical form

Replace $R_2 \to R_2 - 3R_1$

and $R_3 \to R_3 - 4R_1$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 3-3\times1 & 4-3\times2 & 5-3\times3 \\ 4-4\times1 & 6-4\times2 & 8-4\times3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix}$

Again $R_3 \to R_3 - R_2$ $\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0-0 & -2-(-2) & -4-(-4) \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

So the submatrix of A is $\begin{bmatrix} 2 & 3 \\ -2 & -4 \end{bmatrix}$

$$= \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix} = -8+6 = -2 \neq 0$$

So the rank of matrix is 2.

Ex:-2 Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}_{3\times 3}$

Soln

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}_{3\times 3} \begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 \end{matrix}$$

$r(A) \leq (3,3)$

Replace $R_2 \to R_2 - R_1$

& $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 1-1 & 4-2 & 2-3 \\ 2-2\times 1 & 6-2\times 2 & 5-2\times 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Again Replace $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2-2 & -1-(-1) \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{(canonical form)}$$

submatrix of the A is $\begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix}_{2\times 2}$.

$= \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = 2\times(-1) - 2\times 3 = -2-6 = -8 \neq 0$

So the rank of matrix is '2'.

Ex:- Find the rank of Matrix of $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}_{3\times 4}$

Soln :- $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$ 3×4

So $r(A) \leq (3,4)$

Replace $R_2 \to R_2 - 3R_1$ & $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3-3\times 1 & 9-3\times 3 & 12-3\times 4 & 3-3\times 3 \\ 1-1 & 3-3 & 4-4 & 1-3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

Again $R_3 \to R_3 - \frac{1}{3}R_2$

$\left[-2-\frac{1}{3}\times(-6)(-2) \; -2+(2)=0\right]$

$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2-\frac{1}{3}\times(-6) \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (canonical form)

(3)

So the submatrix of A is $\begin{bmatrix} 4 & 3 \\ 0 & -6 \end{bmatrix}_{2\times 2}$

$$= \begin{vmatrix} 4 & 3 \\ 0 & -6 \end{vmatrix} = 4\times(-6) - 0\times 3$$

$$= -24 \neq 0$$

So the rank of Matrix A is '2'.

Ex:- Find the rank of Matrix, $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}_{3\times 3}$

Soln :- $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}_{3\times 3}$

So $r(A) \leq (3, 3)$

Replace $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ -6+2\times 3 & 2+2\times(-1) & 4+2\times 2 \\ -3+3 & 1+(-1) & 2+(2) \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

Again $R_3 \to R_3 - \frac{1}{2} R_2$, $\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4-\frac{1}{2}\times 8 \end{bmatrix}$

CH-03 Differential Equation

26.08.2020

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_{n-1} \frac{dy}{dx} + P_n y = 0$$

where $P_i = \frac{a_i}{a_0}$, $i = 0, 1, 2, \cdots, n$.

$$y = C.F + P.I$$

Rules for finding the complementary Function (C.F):-

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = 0 \quad \text{— (I)}$$

$$\Rightarrow D^n y + P_1 D^{n-1} y + \cdots + P_n y = 0 \quad \left[\because D = \frac{d}{dx}\right]$$

$$\Rightarrow \left[D^n + P_1 D^{n-1} + \cdots + P_n\right] y = 0$$

$$\Rightarrow \boxed{D^n + P_1 D^{n-1} + \cdots + P_n = 0} \quad \text{— (II)}$$

which is known as Auxiliary eqn.

Let $y = e^{mx}$, then

$$Dy = \frac{dy}{dx} = \frac{d}{dx}(e^{mx}) = m e^{mx}$$

$$D^2 y = \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)(m e^{mx}) = m^2 e^{mx}$$

$$D^3 y = \frac{d^3 y}{dx^3} = \frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right) = \frac{d}{dx}(m^2 e^{mx}) = m^3 e^{mx}$$

$$\vdots$$

$$D^{n-1} y = m^{n-1} e^{mx}$$

$$D^n y = m^n e^{mx}$$

Putting these in eqn (II) we have

$$m^n e^{mx} + P_1 m^{n-1} e^{mx} + \cdots + P_n = 0$$

$$\Rightarrow m^n + P_1 m^{n-1} + \cdots + P_n = 0$$

If 'm' is the root of the eqn s, then

$D = m_1, m_2, m_3, \dots m_n$ be the roots of the Auxiliary eqn.

<u>case:-1</u> <u>The roots of A.E are all real and different :-</u>

The roots $m_1, m_2, \dots m_n$ of A.E can be real & different then the eqn can be written in form

$(D-m_1)(D-m_2)(D-m_3)\cdots(D-m_n)\,y = 0$

$\Rightarrow (D-m_1)y = 0 \;,\; (D-m_2)y = 0 \cdots (D-m_n)y = 0$

From we have $(D-m_1)\,y = 0$

$\Rightarrow Dy - m_1 y = 0$

$\Rightarrow \frac{dy}{dx} - m_1 y = 0 \quad \left[\because \frac{dy}{dx} + Py = Q\right]$

$P = -m_1 \;,\; Q = 0$

$\therefore I.F = e^{\int P dx} = e^{\int -m_1 dx} = e^{-\int m_1 dx} = e^{-m_1 x}$

Then the soln is $y \cdot I.f = \int Q \cdot I.f \, dx$

$\Rightarrow y \cdot e^{-m_1 x} = \int 0 \cdot e^{-m_1 x} dx = \int 0\, dx = c_1$

$\Rightarrow y \cdot e^{-m_1 x} = c_1$

$\Rightarrow y = c_1 e^{m_1 x}$

similarly other factors are

$y = c_2 e^{m_2 x}$, $y = c_3 e^{m_3 x}$, ... $y = c_n e^{m_n x}$

So the complete solⁿ (or) complementary function (C.F) is

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}} \quad \text{(iii)}$$

Case :- (ii) The two of the roots of A.E are equal :-

Let $m_1 = m_2$

then the solⁿ be $y = (c_1 + c_2 x) e^{m_1 x} + \cdots + c_n e^{m_n x}$

Ex:-1 solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Solⁿ :- $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

$\Rightarrow D^2 y - 5Dy + 6y = 0 \quad \left[\because D = \frac{d}{dx}\right]$

$\Rightarrow \underbrace{(D^2 - 5D + 6)}_{A.E} y = 0$

Now the auxiliary Eqⁿ is $D^2 - 5D + 6 = 0$

$\Rightarrow D^2 - 2D - 3D + 6 = 0$

$\Rightarrow D(D-2) - 3(D-2) = 0 \Rightarrow (D-2)(D-3) = 0$

Either $D - 2 = 0$ (or) $D - 3 = 0$

$\Rightarrow D = 2$ $\qquad \Rightarrow D = 3$

$\Rightarrow D = 2, 3$ $\qquad \because D = m_1, m_2, m_3 \ldots m_n$

$\Rightarrow D = m_1, m_2$

So the C.F $= c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow$ C.F $= c_1 e^{2x} + c_2 e^{3x}$

Hence the general soln is

$y = c_1 e^{2x} + c_2 e^{3x}$ Ans.

Ex:-2 $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

Soln :- $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

$\Rightarrow D^3y - 3D^2y + 3Dy - y = 0$

$\Rightarrow \underbrace{(D^3 - 3D^2 + 3D - 1)}_{A.E} y = 0$

So the A.E is $D^3 - 3D^2 + 3D - 1 = 0$

$\Rightarrow D^3 - 3\cdot D^2\cdot 1 + 3\cdot D\cdot 1^2 - 1^3 = 0$

$\Rightarrow (D-1)^3 = 0 \Rightarrow (D-1)(D-1)(D-1) = 0$

$\Rightarrow D-1 = 0, \quad D-1 = 0, \quad D-1 = 0$

$\Rightarrow D = 1, \quad \Rightarrow D = 1, \quad \Rightarrow D = 1$

$\Rightarrow D = 1, 1, 1 \quad (m_1, m_2, m_3)$

So Here the roots are equal. So the
C.F $= (c_1 + c_2 x + c_3 x^2) e^{mx}$

$= (c_1 + c_2 x + c_3 x^2) e^{x}$

So the soln is $y = (c_1 + c_2 x + c_3 x^2) e^{x}$ Ans

Q:- $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$

Q:- $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Q:- $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

(5)

26.08.2020

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_{n-1}\frac{dy}{dx} + P_n y = 0$$

$$\Rightarrow D^n y + P_1 D^{n-1}y + P_2 D^{n-2}y + \cdots + P_{n-1}D y + P_n y = 0$$

$$\Rightarrow \underbrace{\left[D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_{n-1}D + P_n\right]}_{A.E}y = 0$$

$$P_i = \frac{a_i}{a_0}, \quad i = 0, 1, 2, \cdots n.$$

$$\Rightarrow D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_{n-1}D + P_n = 0$$

$D = m_1, m_2, m_3 \cdots m_n$ (roots)

(i) Roots are real & different :-

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

(ii) Two roots are equal.

$m_1 = m_2$.

$$y = (c_1 + c_2 x)e^{m_1 x} + \cdots + c_n e^{m_n x}$$

(5)

Ex:- ① Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Soln:- $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ $\left[D = \frac{d}{dx}\right]$

$\Rightarrow D^2y - 5Dy + 6y = 0$

$\Rightarrow \underbrace{[D^2 - 5D + 6]}_{A\cdot E} y = 0$

Now the Auxiliary eqn is

$D^2 - 5D + 6 = 0$

$\Rightarrow D^2 - 2D - 3D + 6 = 0$

$\Rightarrow D(D-2) - 3(D-2) = 0$

$\Rightarrow (D-2)(D-3) = 0$

$\Rightarrow$ Either $D-2=0$ (or) $D-3=0$

$\Rightarrow D = 2$ $\Rightarrow D = 3$

$\Rightarrow D = 2, 3$ $[\because D = m_1, m_2, m_3, \cdots m_n]$

$\Rightarrow D = m_1, m_2 = 2, 3$

Here the roots are real and different

so $C.F = c_1e^{m_1x} + c_2e^{m_2x}$

$\Rightarrow C.F = c_1e^{2x} + c_2e^{3x}$

Hence the General soln is

$y = c_1e^{2x} + c_2e^{3x}$ Ans

Ex:-2 solve $\frac{d^2y}{dx^2}+7\frac{dy}{dx}+12y=0$

Soln:- $\frac{d^2y}{dx^2}+7\frac{dy}{dx}+12y=0$

$\Rightarrow D^2y+7Dy+12y=0$

$\Rightarrow \underbrace{[D^2+7D+12]}_{A.E}y=0$

Now the Auxiliary Eqn is

$D^2+7D+12=0$

$\Rightarrow D^2+3D+4D+12=0$

$\Rightarrow D(D+3)+4(D+3)=0$

$\Rightarrow (D+3)(D+4)=0$

Either $D+3=0$ (or) $D+4=0$

$\Rightarrow D=-3$ (or) $\Rightarrow D=-4$

$\Rightarrow D=-3,-4=m_1,m_2$

Here the roots are Real and different.

So $C.F = c_1e^{m_1x}+c_2e^{m_2x}$

$\Rightarrow C.F = c_1e^{-3x}+c_2e^{-4x}$

Hence the general soln is

$y=c_1e^{-3x}+c_2e^{-4x}$ Ans

Ex:- 3 Solve:- $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

$\left[\begin{matrix} f(x) = 0 & L.E \\ f(x) \neq 0 & N.L.E \end{matrix}\right]$

Soln:- $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

$\Rightarrow D^3y - 3D^2y + 3Dy - y = 0$ $\quad [y = C.F + P.I]$

$\Rightarrow [\underbrace{D^3 - 3D^2 + 3D - 1}_{A.E}]\, y = 0$

Now the Auxiliary eqn is

$D^3 - 3D^2 + 3D - 1 = 0$

$\Rightarrow D^3 - 3\cdot D^2\cdot 1 + 3\cdot D\cdot 1^2 - 1^3 = 0$

$\Rightarrow (D-1)^3 = 0$

$\Rightarrow (D-1)(D-1)(D-1) = 0$

so $D-1 = 0$, $D-1 = 0$, $D-1 = 0$

$\Rightarrow D = 1$, $\Rightarrow D = 1$, $\Rightarrow D = 1$

$\Rightarrow D = 1, 1, 1 = m_1, m_2, m_3$

Here the roots are real and Equal.

so $C.F = (c_1 + c_2x + c_3x^2)e^{mx}$

$\Rightarrow C.F = [c_1 + c_2x + c_3x^2]\, e^{x}$

Hence the general soln is

$y = [c_1 + c_2x + c_3x^2]\, e^{x}$ Ans

Ex:- Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Soln:- $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

$\Rightarrow D^2y - 3Dy + 2y = 0$

$\Rightarrow \underbrace{[D^2 - 3D + 2]}_{A.E} y = 0$

Now the Auxiliary Eqn is

$D^2 - 3D + 2 = 0$

$\Rightarrow D^2 - 2D - D + 2 = 0$

$\Rightarrow D(D-2) - 1(D-2) = 0$

$\Rightarrow (D-2)(D-1) = 0$

Either $D-1=0$ (or) $D-2=0$

$\Rightarrow D=1$ (or) $D=2$

$\Rightarrow D = 1, 2 = m_1, m_2$

So the roots are real & different.

Hence $C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow C.F = c_1 e^{x} + c_2 e^{2x}$

Hence the general soln is

$y = c_1 e^{x} + c_2 e^{2x}$ Ans

H.W :- Q:- $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

Q:- $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$

Soln:- $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$

$\Rightarrow D^2y + 7Dy + 12y = 0$

$\Rightarrow \underbrace{[D^2 + 7D + 12]}_{A.E} y = 0$

Now the Auxiliary Eqns is

$D^2 + 7D + 12 = 0$

$\Rightarrow D^2 + 3D + 4D + 12 = 0$

$\Rightarrow D(D+3) + 4(D+3) = 0$

$\Rightarrow (D+3)(D+4) = 0$

Either $D+3=0$ (or) $D+4=0$

$\Rightarrow D=-3$ (or) $\Rightarrow D=-4$

$\Rightarrow D = -3, -4, = m_1, m_2$

So the $C.F = c_1e^{m_1x} + c_2e^{m_2x}$

$\Rightarrow C.F = c_1e^{-3x} + c_2e^{-4x}$

Hence the General soln is

$y = c_1e^{-3x} + c_2e^{-4x}$ Ans

Case:- III Two of the roots in A·E are complex and different:-

$D = m_1, m_2, m_3, \dots, m_n$

Let the two roots are $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

So the general soln is

$$y = e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

$$\Rightarrow y = e^{\alpha x}\left[\underbrace{(c_1+c_2)}_{A}\cos\beta x + \underbrace{i(c_1-c_2)}_{B}\sin\beta x\right] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

[Euler's function:- $e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$]

$$\Rightarrow y = e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

Case:- IV Two of the roots of A·E are complex and Equal:-

Let $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$

So the soln is

$$y = e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4)\sin\beta x\right] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

Ex:- Solve $\frac{d^3y}{dx^3}+y=0$

Soln:- $\frac{d^3y}{dx^3}+y=0$

$\Rightarrow D^3y+y=0$

$\Rightarrow \underbrace{(D^3+1)}_{A\cdot E}y=0$

Now the Auxiliary Eqn is

$D^3+1=0$

$\Rightarrow (D)^3+(1)^3=0$

$\Rightarrow (D+1)(D^2-D\cdot 1+1^2)=0$

$\Rightarrow (D+1)(D^2-D+1)=0$

Either $D+1=0$ (or) $D^2-D+1=0$

$\Rightarrow D=-1$ | $\Rightarrow D=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow D=\frac{-(-1)\pm\sqrt{(-1)^2-4\cdot 1\cdot 1}}{2\cdot 1}$

$\Rightarrow D=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm\sqrt{3}i}{2}$ $\left[\sqrt{-3}=\sqrt{3\times(-1)}=\sqrt{3}i\right]$

$\Rightarrow D=-1,\ \frac{1+\sqrt{3}i}{2},\ \frac{1-\sqrt{3}i}{2}=m_1,m_2,m_3$

$\alpha=\frac{1}{2},\ \beta=\frac{\sqrt{3}}{2}$; $\frac{1}{2}-\frac{\sqrt{3}i}{2}$

Here the roots are one is real and another two roots are complex and different.

So $C.F=c_1e^{-x}+e^{x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right]$

Hence the general soln is

$y=c_1e^{-x}+e^{x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right]$ Ans

(3) (12)

$\Rightarrow (D^2y + Dy + y)^2 = 0$

Ex:- Solve:- $(D^2+D+1)^2 y = 0$

Soln :- $(\underbrace{D^2+D+1}_{A.E.})^2 y = 0$

Now the A.E is $(D^2+D+1)^2 = 0$

$\Rightarrow (D^2+D+1)(D^2+D+1) = 0$

Either $D^2+D+1 = 0$ (or) $D^2+D+1 = 0$

$\Rightarrow D = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ | $\Rightarrow D = \frac{-1 \pm \sqrt{3}i}{2}$

$= \frac{-1 \pm \sqrt{3}i}{2}$

$\Rightarrow D = \frac{-1 \pm \sqrt{3}i}{2}, \frac{-1 \pm \sqrt{3}i}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i, \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$

α β

$\Rightarrow D = \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$

$= (m_1, m_2, m_3, m_4)$

Here the roots are complex & Equal.

So C.F $= e^{-x/2}\left[(c_1+c_2x)\cos\left(\frac{\sqrt{3}}{2}\right)x + (c_3+c_4x)\sin\left(\frac{\sqrt{3}}{2}\right)x\right]$

Hence the general soln is

$y = e^{-x/2}\left[(c_1+c_2x)\cos\left(\frac{\sqrt{3}}{2}\right)x + (c_3+c_4x)\sin\left(\frac{\sqrt{3}}{2}\right)x\right]$ Ans

(4) (13)

Ex:- $\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0$

$\Rightarrow D^3y+6D^2y+11Dy+6y=0$

$\Rightarrow \underbrace{[D^3+6D^2+11D+6]}_{A\cdot E}y=0$

Now the Auxiliary Equ is

$D^3+6D^2+11D+6=0$

$D=\pm1,\pm2\ldots$

Let $D=1$, $1^3+6\cdot1^2+11\cdot1+6\neq0$

$D=-1$, $(-1)^3+6\cdot(-1)^2+11\cdot(-1)+6$

$=-1+6-11+6=-12+12=0$

So $D=-1$ be any root.

$\Rightarrow D+1=0$

$(D+1)(\ ?\ )=D^3+6D^2+11D+6$

$$
\begin{array}{r|l}
 & D^2+5D+6 \\
D+1 & D^3+6D^2+11D+6 \\
 & \cancel{D^3}+D^2 \\
 & -\quad - \\
\hline
 & 5D^2+11D+6 \\
 & \cancel{5D^2}+5D \\
 & -\quad - \\
\hline
 & 6D+6 \\
 & 6D+6 \\
 & -\quad - \\
\hline
 & 0
\end{array}
$$

So the equ can be written inform

$(D+1)(D^2+5D+6)=0$

$\Rightarrow (D+1)(D^2+3D+2D+6)=0$

$\Rightarrow (D+1)[D(D+3)+2(D+3)]=0$

$\Rightarrow (D+1)(D+2)(D+3)=0$

$\Rightarrow D+1=0, \quad D+2=0, \quad D+3=0$

$\Rightarrow D=-1, \Rightarrow D=-2, \Rightarrow D=-3$

$\Rightarrow D=-1, -2, -3$
$\quad m_1, m_2, m_3$

Here the roots are real & different.

So the C.F $= c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

$\Rightarrow C.F = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

Hence the general soln is

$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$ Ans

[$y =$ C.F (H.L.D.E)
$y =$ C.F + P.I (H.N.L.D.E)]

Ex:-2 Solve: $4D^4 - 4D^3 - 23D^2 + 12D + 36 = 0$

Soln put $D = \pm 1, \pm 2, \pm 3, \ldots$

$D=1, \quad 4-4-23+12+36 \neq 0$

$D=-1 \quad 4+4-23+12+36 \neq 0$

$D=2 \quad 64-32-92+24+36 = 0$

$\Rightarrow D-2=0$

$(D-2)(\ ?\ ) = 4D^4 - 4D^3 - 23D^2 + 12D + 36$

$$
\begin{array}{r|l}
 & 4D^3+4D^2-15D-18 \\ \hline
D-2 & 4D^4-4D^3-23D^2+12D+36 \\
 & 4D^4-8D^3 \\
 & -\quad + \\ \hline
 & 4D^3-23D^2+12D+36 \\
 & 4D^3-8D^2 \\
 & -\quad + \\ \hline
 & -15D^2+12D+36 \\
 & -15D^2+30D \\
 & +\quad - \\ \hline
 & -18D+36 \\
 & -18D+36 \\
 & +\quad - \\ \hline
 & 0
\end{array}
$$

$\Rightarrow (D-2)(4D^3+4D^2-15D-18)=0$ (1)

$4D^3+4D^2-15D-18=0$

Put $D=2$, $32+16-30-18$

$\Rightarrow 48-48=0$

$\Rightarrow D-2=0$

$(D-2)(\,?\,)=4D^3+4D^2-15D-18$

$$
\begin{array}{r|l}
 & 4D^2+12D+9 \\ \hline
D-2 & 4D^3+4D^2-15D-18 \\
 & 4D^3-8D^2 \\
 & -\quad + \\ \hline
 & 12D^2-15D-18 \\
 & 12D^2-24D \\
 & -\quad + \\ \hline
 & 9D-18 \\
 & 9D-18 \\
 & -\quad + \\ \hline
 & 0
\end{array}
$$

(3)

$(D-2)(4D^2+12D+9)=0$

$\Rightarrow (D-2)\left[(2D)^2+2\cdot 2D\cdot 3+3^2\right]=0$

$\Rightarrow (D-2)\left[(2D+3)^2\right]=0$ — (ii)

Now equating (i) & (ii) we have,

$(D-2)(D-2)(2D+3)^2=0$

$\Rightarrow (D-2)(D-2)(2D+3)(2D+3)=0$

$\Rightarrow D-2=0\ ,\ D-2=0\ ,\ 2D+3=0\ ,\ 2D+3=0$

$\Rightarrow D=2\ ,\ \Rightarrow D=2\ ,\ \Rightarrow D=-3/2\ ,\ \Rightarrow D=-3/2$

$\Rightarrow D=2,2,-3/2,-3/2$
m_1, m_2, m_3, m_4

Here the roots are real & equal

So the C.F $=(c_1+c_2x)e^{2x}+(c_3+c_4x)e^{-3/2x}$

Hence the general soln is

$y=(c_1+c_2x)e^{2x}+(c_3+c_4x)e^{-3/2x}$ Ans

17

Q:- $\frac{d^3y}{dx^3} - \frac{dy}{dx} = 0$

(2) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$

(3) $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

(4) $\frac{d^2x}{dt^2} + 3a\frac{dx}{dt} - 4a^2x = 0$

Partial Differential Equation :-

A partial diff. eqn is a relation betn the independent variables, dependent variable and it's partial derivatives. If $z = f(x,y)$, z is called a func of two independent variables x & y. The partial derivatives.

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial y^2} = t$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s$$

An equation involving one or more partial derivatives is called a partial differentiation. The order of a partial diff. eqn (P.D.E.) is the order of the heighest partial derivative in the eqn & it's degree is deg. of this derivative.

Ex:- Form the P.D.E.

(1) Solve :- $z = ax + by + a^2 + b^2$

Soln :- $z = ax + by + a^2 + b^2$ — (1)

Differentiating partially eqn (1) w.r.t. x

$\frac{\partial z}{\partial x} = a$ — (2)

Again diff. eqn (1) w.r.t. y

$\frac{\partial z}{\partial y} = b$ — (3)

Hence the eqn (1) can be written as

$z = \frac{\partial z}{\partial x} x + \frac{\partial z}{\partial y} y + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$

$\Rightarrow z = px + qy + p^2 + q^2$ $\boxed{\because p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}}$

Ans

(2) Solve :- $(x-a)^2 + (y-b)^2 + z^2 = c^2$

Soln :- $(x-a)^2 + (y-b)^2 + z^2 = c^2$ — (1)

Diff. eqn (1) partially w.r.t. 'x'

$\frac{\partial z}{\partial x} \Rightarrow 2(x-a) + 0 + 2z\frac{\partial z}{\partial x} = 0$

$\Rightarrow (x-a) + z\frac{\partial z}{\partial x} = 0$

$\Rightarrow a = x + z\frac{\partial z}{\partial x}$ — (2)

Again diff. eqn (1) partially w.r.t. 'y'

$0 + 2(y-b) + 2z\frac{\partial z}{\partial y} = 0 \Rightarrow y - b + z\frac{\partial z}{\partial y} = 0$

$\Rightarrow b = y + z\frac{\partial z}{\partial y}$ — (3)

Hence eqn (1) can be written in form

$$\left(x-x-z\frac{\partial z}{\partial x}\right)^2+\left(y-y-z\frac{\partial z}{\partial y}\right)^2+z^2=c^2$$

$$\Rightarrow z^2\left(\frac{\partial z}{\partial x}\right)^2+z^2\left(\frac{\partial z}{\partial y}\right)^2+z^2=c^2$$

$$\Rightarrow z^2(p^2+q^2+1)=c^2 \text{ Ans}$$

Ex:- Solve $z=f(x^2-y^2)$

Soln $z=f(x^2-y^2)$ — (1)

Diff. eqn (1) partially w.r.t. x

$$\frac{\partial z}{\partial x}=f'(x^2-y^2)\cdot 2x \quad \text{— (2)}$$

Again eqn (2) by partially w.r.t. y

$$\frac{\partial z}{\partial y}=f'(x^2-y^2)(-2y) \quad \text{— (3)}$$

Dividing eqn (2) by eqn (3) we have

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}=\frac{\cancel{f'(x^2-y^2)}(2x)}{\cancel{f'(x^2-y^2)}(-2y)}=-\frac{x}{y}$$

$$\Rightarrow \frac{p}{q}=-\frac{x}{y} \Rightarrow py=-qx$$

$$\boxed{\Rightarrow py+qx=0} \text{ Ans}$$

Ex:- Solve $z=f\left(\frac{xy}{z}\right)$

Linear Equations of the First order :-

A Linear partial diff. eqn of the 1st order is commonly known as Lagrange's linear eqn is of the form

$$Pp + Qq = R$$

where P, Q, R both fun^n of x, y, z

So the general soln of the linear partial diff. eqn $Pp + Qq = R$ — (1)

is $f(u, v) = 0$ — (2)

where 'f' is an arbitrary fun^n & $u(x,y,z) = c_1$ & $v(x,y,z) = c_2$ form the soln of eqn $\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$

Solve :- $pyz + qzx = xy$

Soln :- $pyz + qzx = xy$ — (1)

$P = yz, \; Q = zx, \; R = xy$

It's subsidiary eqns are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

considering 1st two ratio $\frac{dx}{yz} = \frac{dy}{zx}$

$$\Rightarrow x\,dx = y\,dy$$

Taking Integration on both sides we have

$\int x\,dx = \int y\,dy$

$\Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + c_1$ $\boxed{\Rightarrow x^2 - y^2 = 2c_1 = c_1}$ — 1

Again considering 2nd two ratios,

$\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow y\,dy = z\,dz$

Integrate on both sides we have

$\int y\,dy = \int z\,dz \Rightarrow \frac{y^2}{2} = \frac{z^2}{2} + c_2$

$\Rightarrow \boxed{y^2 - z^2 = 2c_2 = c_2}$ — 2

Hence the required solⁿ is

$f(x^2 - y^2, y^2 - z^2) = 0$

Ex:- Solve :- $x(y-z)p + y(z-x)q = z(x-y)$

solⁿ:- where $P = x(y-z)$, $Q = y(z-x)$

$R = z(x-y)$

It's subsidiary eqⁿ's are

$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

Let $P' = 1$, $Q' = 1$, R'

so $PP' + QQ' + RR' = 0$

$\Rightarrow x(y-z) + y(z-x) + z(x-y) = 0$

$\Rightarrow xy - xz + yz - xy + zx - yz = 0$

Hence $P'dx + Q'dy + R'dz = 0$ is integrable

$\Rightarrow 1 \cdot dx + 1 \cdot dy + 1 \cdot dz = 0$

Taking Integration on both sides

$\Rightarrow \int dx + \int dy + \int dz = 0$

$\Rightarrow x + y + z = c$ — (1)

Again let $P'' = \frac{1}{x}$, $Q'' = \frac{1}{y}$, $R'' = \frac{1}{z}$

$\Rightarrow PP'' + QQ'' + RR'' =$ ~~$\frac{1}{x} \cdot x(y-z) + \frac{1}{y} \cdot y(z-x)$~~

$= \frac{1}{\cancel{x}} \cdot \cancel{x}(y-z) + \frac{1}{\cancel{y}} \cdot \cancel{y}(z-x) + \frac{1}{\cancel{z}} \cdot \cancel{z}(x-y)$

$= y - z + z - x + x - y = 0$ is integrable

Taking Integration on both sides

we have $P''dx + Q''dy + R''dz = 0$

$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$\Rightarrow \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = \int 0$

$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = c$ –

$\Rightarrow \ln x + \ln y + \ln z = c$

$\Rightarrow xyz = \ln c = c_2$ — (2)

$\Rightarrow$

So the required solution is

$F(x+y+z, xyz) = 0$ Ans

Ex!- Solve $y^2p - xyq = x(z-2y)$

Sol^n where $P = y^2,\ Q = -xy,\ R = x(z-2y)$

It's subsidiary eq^n is

$$\cancel{\frac{P}{dx} = \frac{Q}{dy} = \frac{R}{dz}} \quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Considering integration of 1st two ratios $\frac{dx}{y\cancel{^2}} = \frac{dy}{-x\cancel{y}}$

$$\Rightarrow \frac{dx}{y} = \frac{dy}{-x} \Rightarrow -x\,dx = y\,dy$$

$$\Rightarrow -\int x\,dx = \int y\,dy$$

$$\Rightarrow -\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

$$\Rightarrow x^2 + y^2 = 2c_1 = c_1 \quad \text{--- (1)}$$

Again $\frac{dy}{-y\cancel{x}} = \frac{dz}{\cancel{x}(z-2y)} \Rightarrow \frac{dy}{-y} = \frac{dz}{z-2y}$

$$\Rightarrow (z-2y)\,dy = -y\,dz$$

$$\Rightarrow z\,dy - 2y\,dy = -y\,dz$$

$$\Rightarrow z\,dy + y\,dz = 2y\,dy$$

$$\Rightarrow d(yz) = 2y\,dy$$

Taking Integration on both sides we have

$$\int d(yz) = \int 2y\,dy$$

$$\Rightarrow yz = \cancel{2}\cdot\frac{y^2}{\cancel{2}} + c_2$$

$$\Rightarrow yz = y^2 + c_2$$

$$\Rightarrow yz - y^2 = c_2 \quad \text{--- (2)}$$

Hence the required soln is

$$F(x^2+y^2,\ yz-y^2) = 0 \quad \underline{\underline{\text{Ans}}}$$

<u>Ex:-</u> $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

<u>Ex:-</u> $x(y^2-z^2)p + y(z^2-x^2)q - (x^2-y^2)z = 0$

— o —

CH-06 Laplace Transformation (L·T)

Gamma function :- The Gamma function is a funcn of single real variable. It is denoted by $\Gamma(n)$ and is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \quad \text{— (i)}$$

where 'n' is a +ve integer.

Recurrence formula :-

(i) $\Gamma(n+1) = n\Gamma(n)$

(ii) $\Gamma(n+1) = n!$ For a +ve integer 'n'.

Note $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

$\Gamma(0) = \dfrac{\Gamma(1)}{0} = \infty$

$\Gamma(-1) = \dfrac{\Gamma(0)}{-1} = \infty$

$\Gamma(-2) = \dfrac{\Gamma(-1)}{-2} = \infty$ and so on.

Again we know $\Gamma(n+1) = n\Gamma(n)$

then $\Gamma(2) = \Gamma(1+1) = 1\Gamma(1)$

$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2\cdot 1 = 2!$

$\Gamma(4) = \Gamma(4+1) = 3\Gamma(3) = 3\cdot 2! = 3!$

$\vdots$

$\Gamma(n+1) = n!$

Formula :-

$$\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

$$\boxed{\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}}$$

Ex:- $\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)$

$= \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$

Laplace Transform of a function $f(t)$

A transformation is a mathematical device which converts one funⁿ to another function.

Given a funⁿ $f(t)$ of a real variable $t>0$ if we multiply it by e^{-st} and integrate w.r.t. t between the limits 0 and ∞, the result is a function of s, say $F(s)$. This funⁿ $F(s)$ is called the Laplace transform of $f(t)$. It is denoted by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

. 's' is a parameter which may real or complex.

Linearity property of L.T :-

If a, b, c be any constants and f, g, h any function of 't', then

$L[af(t) + b\,g(t) - c\,h(t)]$

$= a\,L[f(t)] + b\,L[g(t)] - c\,L[h(t)]$.

L.T of some simple functions :-

(i) Transform of a constant function :-

Let $f(t) = k$ a constant

Then $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} \cdot k\,dt$

$= k\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{k}{s}, \quad s > 0$

If $k = 1$, $L[1] = \frac{1}{s}, \quad s > 0$

(ii) Transform of Algebric function :-

Let $f(t) = t$

Then $L[t] = \int_0^\infty e^{-st} \cdot f(t)\,dt = \int_0^\infty e^{-st} \cdot t \cdot dt$

$= \left[t \cdot \frac{e^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty$

$$= -\frac{1}{s^2}\left(e^{-\infty} - e^{-0}\right) = \frac{1}{s^2}, \; s > 0$$

$$\Rightarrow L[t] = \frac{1}{s^2}, \; s > 0$$

(III) Transform of polynomial Fun^n :-

Let $f(t) = t^2$

then $L[t^2] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} \cdot t^2 \, dt$

$$= \left[t^2 \cdot \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, 2t\, dt$$

$$= 0 + \frac{2}{s}\int_0^\infty e^{-st}\,dt \quad \left[\because \lim_{t\to\infty} t^2 \cdot e^{-st} = 0\right]$$

$$= \frac{2}{s} \cdot L[t] = \frac{2}{s} \cdot \frac{1}{s^2} = \frac{2}{s^3}$$

$$\Rightarrow L[t^2] = \frac{2}{s^3}, \; s > 0$$

(IV) Transform t^n :-

when $f(t) = t^n$

Then $L[t^n] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} \cdot t^n \, dt$ $[n \to +ve]$

$$= \left[t^n \cdot \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \cdot n \cdot t^{n-1}\,dt$$

$$= 0 + \frac{n}{s}\int_0^\infty e^{-st} \cdot t^{n-1}\,dt = \frac{n(n-1)}{s^2} L[t^{n-2}]$$

$$\Rightarrow L[t^n] = \frac{n!}{s^{n+1}}, \; s > 0, \; n = 0, 1, 2 \ldots$$

(V)

Formula:- when 'n' is a +ve Integer

i.e fraction, $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

(VI) **Transform of Exponential func:-**

Let $f(t) = e^{at}$

$$L[e^{at}] = \int_0^\infty e^{-st} \cdot e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}$$

$$\boxed{\Rightarrow L[e^{at}] = \frac{1}{s-a}\ ,\ s>a}$$

Similarly $\boxed{L[e^{-at}] = \frac{1}{s+a}\ ,\ s>-a}$

(VIII) **Transform of Hyperbolic Function:-**

Let $f(t) = \cosh at$

$$\cosh at = \frac{e^{at}+e^{-at}}{2}\ ;\ \sinh at = \frac{e^{at}\cdot e^{-at}}{2}$$

$$L[\cosh at] = \int_0^\infty e^{-st} \cdot \cosh at\,dt$$

$$= \int_0^\infty e^{-st}\left[\frac{e^{at}+e^{-at}}{2}\right]dt = \frac{1}{2}\left[\int_0^\infty e^{-st}\cdot e^{at}\,dt + \int_0^\infty e^{-st}\cdot e^{at}\,dt\right]$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt + \int_0^\infty e^{-(s+a)t}\,dt\right]$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty + \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} = \frac{1}{2}\left\{\frac{s+a+s-a}{(s+a)(s-a)}\right\} = \frac{1}{2}\times\frac{2s}{s^2-a^2}$$

$$\Rightarrow \boxed{L[\cosh at] = \frac{s}{s^2-a^2},\ s>|a|}$$

Similarly $\boxed{L[\sinh at] = \frac{a}{s^2-a^2},\ s>|a|}$

(IX) Transform of Trigonomatric Func:-

Let $f(t) = \cos at, \sin at$

$$e^{iat} = \cos at + i\sin at$$

$$e^{-iat} = \cos at - i\sin at$$

$$\cos at = \frac{e^{iat}+e^{-iat}}{2},\quad \sin at = \frac{e^{iat}-e^{-iat}}{2}$$

Let $f(t) = \sin at$

Then $L[\sin at] = \int_0^\infty e^{-st}\cdot\sin at\,dt = \int_0^\infty e^{-st}\cdot\frac{e^{iat}-e^{-iat}}{2}\,dt$

$$= \frac{1}{2i}\left[\int_0^\infty e^{-(s-ia)t}\,dt - \int_0^\infty e^{-(s+ia)t}\,dt\right]$$

$$= \frac{1}{2i}\left[\frac{e^{-(s-ia)t}}{-(s-ia)} - \frac{e^{-(s+ia)t}}{-(s+ia)}\right]_0^\infty$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{1}{2i}\left[\frac{s+ia-s+ia}{(s-ia)(s+ia)}\right]$$

$$= \frac{1}{2i}\left[\frac{2ia}{s^2+a^2}\right] = \frac{a}{s^2+a^2} \quad (6)$$

$$\Rightarrow \boxed{L[\sin at] = \frac{a}{s^2+a^2},\ s>|a|}$$

Similarly $\boxed{L[\cos at] = \frac{s}{s^2+a^2},\ s>|a|}$

First shifting Theorem:-

If $L[f(t)] = F(s)$, then

$L[e^{at}f(t)] = F(s-a),\ s-a>0$

2nd shifting Theorem:-

If $f(t) = F(s)$, then $g(t) = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$

Then $L[g(t)] = e^{-as}F(s)$.

Formula:- ① $L[e^{at}] = \frac{1}{s-a}$

② $L[1] = \frac{1}{s}$

③ $L[e^{at}t^n] = \frac{n!}{(s-a)^{n+1}},\ n=0,1,2,\ldots$

$= \frac{\Gamma(n+1)}{(s-a)^{n+1}},\ \cancel{n\geq 0}\ s-a>0$

④ $L[e^{at}\cos bt] = \frac{s-a}{(s-a)^2+b^2}$

⑤ $L[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2},\ s-a>0$

(6) $L[e^{at}\cosh bt] = \dfrac{s-a}{(s-a)^2 - b^2}, \; s-a > 0$

(7) $L[e^{at}\sinh bt] = \dfrac{b}{(s-a)^2 - b^2}, \; s-a > 0$

Problems

(1) $L[8] = \dfrac{8}{s}$

(2) $L[e^{4t}] = \dfrac{1}{s-4}$

(3) $L[4e^{5t}] = \dfrac{4}{s-5}$

(4) $L[2\cosh 2t] = \dfrac{2s}{s^2-4}$

(5) $L[3\sinh 3t] = \dfrac{9}{s^2-9}$

Ex:- Find the L.T of following functions

(i) $1 + 2t^3 - 4e^{3t} + 5e^{-t}$

(ii) $3\cosh 4t - 4\sin 3t$

Soln:- (i) $L[1 + 2t^3 - 4e^{3t} + 5e^{-t}]$

$= L[1] + L[2t^3] - L[4e^{3t}] + L[5e^{-t}]$

$= L[1] + 2L[t^3] - 4L[e^{3t}] + 5L[e^{-t}]$

$= \dfrac{1}{s} + 2 \cdot \dfrac{3!}{s^4} - \dfrac{4}{s-3} + \dfrac{5}{s+1}$

$= \dfrac{1}{s} + \dfrac{12}{s^4} - \dfrac{4}{s-3} + \dfrac{5}{s+1}$ Ans

(ii) $L[3\cosh 4t + 4\sin 3t]$

$= L[3\cosh 4t] + L[4\sin 3t]$

$= 3L[\cosh 4t] + 4L[\sin 3t]$

$= 3 \cdot \frac{s}{s^2-4^2} + 4 \cdot \frac{3}{s^2+3^2} = \frac{3s}{s^2-16} + \frac{12}{s^2+9}$ Ans

Ex:- find L.T of $(\sin t - \cos t)^2$

Soln Here $f(t) = (\sin t - \cos t)^2$

$= \sin^2 t + \cos^2 t - 2\sin t \cdot \cos t$

$= 1 - 2\sin t \cdot \cos t = 1 - \sin 2t$

Taking L.T on both sides we have

$L[(\sin t - \cos t)^2] = L[1 - \sin 2t]$

$= L[1] - L[\sin 2t] = \frac{1}{s} - \frac{2}{s^2+4}$

$\Rightarrow L[(\sin t - \cos t)^2] = \frac{s^2+4-2s}{s(s^2+4)}$ Ans

Ex:- find $L[\sin^2 3t]$

Soln Here $f(t) = \sin^2 3t = \frac{1-\cos 6t}{2}$

$= \frac{1}{2} - \frac{\cos 6t}{2}$

Taking L.T on both sides we have

$$L[\sin^2 3t] = L\left[\frac{1}{2} - \frac{\cos 6t}{2}\right] = L\left[\frac{1}{2}\right] - L\left[\frac{\cos 6t}{2}\right]$$

$$= \frac{1}{2s} - \frac{1}{2} \times \frac{s}{s^2+6^2} = \frac{1}{2s} - \frac{1}{2(s^2+36)}$$

$$\Rightarrow L[\sin^2 3t] = \frac{2s^2+36-s^2}{2s(s^2+36)} = \frac{18}{s(s^2+36)} \text{ Ans}$$

Ex:- Find $L[\cos^3 2t]$

Soln :- Here $f(t) = \cos^3 2t$

We know $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow 4\cos^3\theta = \cos 3\theta - 3\cos\theta$

$\Rightarrow \cos^3\theta = \dfrac{\cos 3\theta - 3\cos\theta}{4}$

Hence $\cos^3 2t = \frac{1}{4}[\cos 6t + 3\cos 2t]$

Taking L-T on both sides, we have

$$L[\cos^3 2t] = \frac{1}{4} L[\cos 6t + 3\cos 2t]$$

$$= \frac{1}{4}\left\{L[\cos 6t] + L[3\cos 2t]\right\} = \frac{1}{4}\left\{\frac{s}{s^2+36} + 3\cdot\frac{s}{s^2+4}\right\}$$

$$= \frac{1}{4}\left\{\frac{s^2+4s+3s^3+108s}{(s^2+36)(s^2+4)}\right\} = \frac{1}{4}\left\{\frac{4s^3+112s}{(s^2+36)(s^2+4)}\right\}$$

$$= \frac{1}{4} \times 4\,\frac{s(s^2+28)}{(s^2+36)(s^2+4)} = \frac{s(s^2+28)}{(s^2+36)(s^2+4)} \text{ Ans}$$

Ex:- Find $L[f(t)]$ if $f(t) = \begin{cases} e^t, & 0<t<1 \\ 0, & t>1 \end{cases}$

Sol^n $f(t) = \begin{cases} e^t, & 0<t<1 \\ 0, & t>1 \end{cases}$

Then $L[f(t)] = \int_0^1 e^{-st} \cdot e^t dt + \int_1^\infty e^{-st} \cdot 0 \cdot dt$

$= \int_0^1 e^{-st+t} dt = \int_0^1 e^{t(1-s)} dt = \frac{1}{1-s} \left[e^{(1-s)t}\right]_0^1$

$= \frac{1}{1-s} \times \left\{e^{(1-s)} - 1\right\}$ Ans

Transform of derivatives, integrals multiplication by t^n and division by 't':-

L.T of a function $f(t)$ is known as we can deduce the L.T of (i) $e^{at} f(t)$ where 'a' is +ve or -ve constant. (ii) $t^n f(t)$, where 'n' is a +ve integer and (iii) $(1/t) f(t)$ by using certain standard rules.

(1) Transform of $e^{at} \cdot f(t)$:

If $L[f(t)] = F(s)$, then for a +ve or -ve const. 'a', $L[e^{at} f(t)] = F(s-a)$ — (1):

(ii) Transform of $t^n f(t)$

If $L[f(t)] = F(s)$, then for a +ve integer 'n', $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$ — (2)

(iii) Transform of $\frac{1}{t} f(t)$ (division by t)

If $L[f(t)] = F(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

Ex:- (1) Find L.T of $e^{-t}(3\sinh 2t - 2\cosh 3t)$

Soln:- $L[3\sinh 2t - 2\cosh 3t]$

$= L[3\sinh 2t] - L[2\cosh 3t]$

$= 3L[\sinh 2t] - 2L[\cosh 3t]$

$= 3\left(\frac{2}{s^2-2^2}\right) - 2\left(\frac{s}{s^2-3^2}\right)$

By using shifting rule, we have

$L[e^{-t}(3\sinh 2t - 2\cosh 3t)] = \frac{6}{(s+1)^2-2^2} - \frac{2(s+1)}{(s+1)^2-3^2}$

Ans

(2) Find L.T of $e^{3t}\sin^2 t$

Soln:- $\sin^2 t = \frac{1}{2}(1-\cos 2t)$

$\Rightarrow L(\sin^2 t) = \frac{1}{2} L(1-\cos 2t) = \frac{1}{2}[L(1) - L(\cos 2t)]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

By shifting property

$$L[e^{3t}\sin^2 t] = \frac{1}{2}\left[\frac{1}{s-3} - \frac{s-3}{(s-3)^2+4}\right]$$ Ans

Ex:- (3) Find L.T of $[e^{-t}\sin 4t + t\cos 2t]$

Soln:- $L[\sin 4t] = \dfrac{4}{s^2+16}$

By shifting property,

$$[e^{-t}\sin 4t] = \frac{4}{(s+1)^2+16} \quad \text{--- (1)}$$

Again $L[\cos 2t] = \dfrac{s}{s^2+4}$,

By shifting property.

$$L[t\cos 2t] = -\frac{d}{ds}\left[\frac{s}{s^2+4}\right] = \frac{s^2-4}{(s^2+4)^2} \quad \text{--- (11)}$$

Adding (1) & (11) we get

$$L[e^{-t}\sin 4t + t\cos 2t] = \frac{4}{(s+1)^2+16} + \frac{s^2-4}{(s^2+4)^2}$$

Ans

Ex:- (4) Find $L[te^{-t}\cosh t]$

Soln Here $f(t) = \cosh t$

$$\Rightarrow F(s) = \frac{s}{s^2-1}$$

But $L[e^{-t}\cdot f(t)] = \dfrac{s+1}{(s+1)^2-1}$

Hence $L\{t(e^{-t}f(t))\} = (-1)\frac{d}{ds}F(s+1)$

$$= -\frac{d}{ds}\left\{\frac{s+1}{(s+1)^2-1}\right\} = -\left\{\frac{s^2+2s+1-1-(s+1)2\cdot(s+1)}{(s^2+2s+1-1)^2}\right\}$$

$$= -\frac{(s^2+2s-2s^2-4s-2)}{(s^2+2s)^2} = \frac{s^2+2s+2}{s^2(s+2)^2}$$

Ex: ⑤ Find $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$

Soln Here $f(t) = e^{-at}-e^{-bt}$.

$$\Rightarrow F(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

we have $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right)ds$

$$= \int_s^\infty \frac{1}{s+a}ds - \int_s^\infty \frac{1}{s+b}ds = \left[\log(s+a)\right]_s^\infty - \left[\log(s+b)\right]_s^\infty$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty = \left[\log\frac{(s+a)}{(s+b)}\right]_s^\infty = \log\frac{s+a}{s+b}$$

Ans

Ex: ⑥ Find L.T. of ① $\frac{\sin at}{t}$, ⑪ $\frac{\cos at - \cos bt}{t}$

Soln ① we have $L[\sin at] = \frac{a}{s^2+a^2}$

so $L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2}ds = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$ Ans

Sol^{n} (7) We have $L[\cos at - \cos bt]$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

so $$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty = \frac{1}{2}\left[\lim_{s\to\infty}\log\frac{s^2+a^2}{s^2+b^2} - \log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\left[\lim_{s\to\infty}\log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right) - \log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\left[0 - \log\frac{s^2+a^2}{s^2+b^2}\right] = -\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$= \log\left(\frac{s^2+a^2}{s^2+b^2}\right)^{-1/2}$$

Ans

Inverse Laplace Transform :-

If $F(s)$ is the laplace transform of a function $f(t)$. i.e $L\{f(t)\} = F(s)$ Then $f(t)$ is called the inverse laplace transform and written as

$$f(t) = L^{-1}\{F(s)\}$$

where L^{-1} denote the inverse laplace transform.

Then $L[f(t)] = F(s) = L^{-1}F(s) = f(t)$

Formula

① $L^{-1}\left[\frac{1}{s}\right] = 1$

③ $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

② $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

④ $L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$

$n = +ve$,

⑤ $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

⑧ $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

⑥ $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

⑦ $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

⑨ $L^{-1}\left\{\frac{(s+b)}{(s+b)^2+a^2}\right\} = e^{-bt}\cos at$

⑩ $L^{-1}\left\{\frac{1}{(s+b)^2+a^2}\right\} = \frac{1}{a}e^{-bt}\sin at$

L.T by Method of Partial fraction :-

① A fraction form is $\frac{P(s)}{Q(s)}$

A fraction is said to be proper if the degree of numerator is less than degree of denominator.

→ Check whether the given fraction is a proper fraction or not.

So →(i) If the denominator is linear but not repeated which is of the form

$$\frac{A}{s-a}+\frac{B}{s-b}+\frac{C}{s-c}+\cdots$$

To find unknown A, put $x-a=0$
$\Rightarrow x=a$

→(ii) If the denominator is linear but repeated :

$$\frac{A}{s-a}+\frac{B}{(s-a)^2}+\frac{C}{(s-a)^3}+\cdots$$

→(iii) When the denominator is quadratic but not repeated

$$\frac{As+B}{s^2+\alpha}+\frac{Cs+D}{s^2+\beta}+\frac{Es+F}{s^2+\gamma}+\cdots$$

→(iv) When the denominator is quadratic but repeated

$$\frac{As+B}{s^2+\alpha}+\frac{Cs+D}{(s^2+\alpha)^2}+\frac{Es+F}{s^2+\beta}+\cdots$$

Ex:- (1) Find Inverse laplace of

(i) $\frac{3}{s+3}$ (ii) $\frac{2s}{s^2+9}$

Soln:- $L[e^{at}]=\frac{1}{s-a} \Rightarrow L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$

For $a=-3$, then $L^{-1}\left[\frac{1}{s+3}\right]=e^{-3t}$

Hence $L^{-1}\left[\frac{3}{s+3}\right]=3L^{-1}\left[\frac{1}{s+3}\right]=3e^{-3t}$

(ii) $L[\cos at] = \dfrac{s}{s^2+a^2}$

$\Rightarrow L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at$

Hence $L^{-1}\left[\dfrac{2s}{s^2+9}\right] = 2L^{-1}\left[\dfrac{s}{s^2+9}\right] = 2\cos 3t$

Ans

(3) Find $L^{-1}\left[\dfrac{3s+7}{s^2-2s-3}\right]$

Soln:- $\dfrac{3s+7}{s^2-2s-3} = \dfrac{3s}{s^2-2s-3} + \dfrac{7}{s^2-2s-3}$

$= \dfrac{3(s-1+1)}{(s-1)^2-2^2} + \dfrac{7}{(s-1)^2-2^2}$

$\Rightarrow \dfrac{3s+7}{s^2-2s-3} = \dfrac{3(s-1)}{(s-1)^2-2^2} + \dfrac{3}{(s-1)^2-2^2} + \dfrac{7}{(s-1)^2-2^2}$

By taking I.L.T we have

$L^{-1}\left[\dfrac{3s+7}{s^2-2s-3}\right] = L^{-1}\left[\dfrac{3(s-1)}{(s-1)^2-2^2}\right] + L^{-1}\left[\dfrac{3}{(s-1)^2-2^2}\right]$

$+ L^{-1}\left[\dfrac{7}{(s-1)^2-2^2}\right]$

$= 3e^t\cosh 2t + \dfrac{3}{2}e^t\sinh 2t + \dfrac{7}{2}e^t\sinh 2t$

$= 3e^t\cosh 2t + 5e^t\sinh 2t$

$= 3e^t\left[\dfrac{e^{2t}+e^{-2t}}{2}\right] + 5e^t\left[\dfrac{e^{2t}-e^{-2t}}{2}\right]$

$= \dfrac{3}{2}(e^{3t}+e^{-t}) + \dfrac{5}{2}(e^{3t}-e^{-t})$

$= \dfrac{1}{2}[3e^{3t}+3e^{-t}+5e^{3t}-5e^{-t})$

$= \dfrac{1}{2}(8e^{3t}-2e^{-t}) = \dfrac{1}{\cancel{2}}\times\cancel{2}(4e^{3t}-e^{-t}) = 4e^{3t}-e^{-t}$

Ans

Ex:- Find $L^{-1}\left[\frac{s^2+s-2}{s(s+3)(s-2)}\right]$

Soln $\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$

$\Rightarrow \frac{s^2+s-2}{\cancel{s(s+3)(s-2)}} = \frac{A(s+3)(s-2) + B\cdot s(s-2) + C\cdot s(s+3)}{\cancel{s(s+3)(s-2)}}$

$\Rightarrow s^2+s-2 = A(s+3)(s-2) + B\cdot s(s-2) + C\cdot s(s+3)$ —(1)

putting $s+3=0 \Rightarrow \boxed{s=-3}$ in eqn (1)
we have

$9-3-2 = A\underbrace{(-3+3)}_{0}(-3-2) + B\cdot(-3)(-3-2) + C\cdot(-3)\underbrace{(-3+3)}_{0}$

$\Rightarrow 4 = 15B \quad \boxed{\Rightarrow B = \frac{4}{15}}$

Again putting $s=0$ in eqn (1) we have

$-2 = A(0+3)(0-2) + B\cdot 0 + C\cdot 0$

$\Rightarrow -2 = -6A \Rightarrow A = \frac{-2}{-6} \Rightarrow \boxed{A = \frac{1}{3}}$

Putting $s-2=0 \Rightarrow s=2$ in eqn (1) we have

$4+2-2 = A\cdot 0 + B\cdot 0 + C\cdot 2\cdot(2+3)$

$\Rightarrow 4 = 10C$

$\Rightarrow C = \frac{4}{10} \Rightarrow \boxed{C = \frac{2}{5}}$

Putting the value of A, B, C in eqn ①

we have

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{1/3}{s} + \frac{4/15}{s+3} + \frac{2/5}{s-2}$$

By taking I.L.T, we have

$$L^{-1}\left[\frac{s^2+s-2}{s(s+3)(s-2)}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{s}\right] + \frac{4}{15}L^{-1}\left[\frac{1}{s+3}\right] + \frac{2}{5}L^{-1}\left[\frac{1}{s-2}\right]$$

$$= \frac{1}{3}\times 1 + \frac{4}{15}\times e^{-3t} + \frac{2}{5}\times e^{2t}$$

$$= \frac{1}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$$ Ans

Ex:- Find $L^{-1}\left[\frac{s}{(s-3)(s^2+4)}\right]$

Soln:- $$\frac{s}{(s-3)(s^2+4)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow \frac{s}{(s-3)(s^2+4)} = \frac{A(s^2+4)+(Bs+C)(s-3)}{(s-3)(s^2+4)}$$

$$\Rightarrow s = A(s^2+4) + (Bs+C)(s-3) \quad \text{— ①}$$

Putting $s-3=0 \Rightarrow \boxed{s=3}$ in eqn ① we have

$$3 = A(3^2+4) + (Bs+C)\times 0$$

$$\Rightarrow 3 = 13A \Rightarrow \boxed{A = 3/13}$$

The eqn (1) can be written in form

$$s = A(s^2+4) + (Bs+C)(s-3)$$

$$\boxed{\Rightarrow s = As^2 + 4A + Bs^2 - 3Bs + Cs - 3C}$$

By equating the co-efficient of (s^2), we get

$$0 = A + B \quad \boxed{\Rightarrow B = -A = -3/13}$$

Again equating the co-efficient of 's'

$$\Rightarrow 1 = -3B + C = -3 \times \frac{-3}{13} + C$$

$$\Rightarrow C = 1 - \frac{9}{13} = \frac{13-9}{13} \Rightarrow \boxed{C = 4/13}$$

Now we put the value of A, B, C in ~~eqn (1)~~ we have

$$\frac{s}{(s-3)(s^2+4)} = \frac{3}{13(s-3)} + \frac{\frac{-3}{13}s + 4/13}{s^2+4}$$

$$\Rightarrow \frac{s}{(s-3)(s^2+4)} = \frac{3}{13(s-3)} - \frac{3}{13} \times \frac{s}{s^2+4} + \frac{4}{13(s^2+4)}$$

Taking I.L.T on both sides we have

$$L^{-1}\left[\frac{s}{(s-3)(s^2+4)}\right] = L^{-1}\left[\frac{3}{13(s-3)}\right] - L^{-1}\left[\frac{3s}{13(s^2+4)}\right] + L^{-1}\left[\frac{4}{13(s^2+4)}\right]$$

$$= \frac{3}{13}e^{3t} - \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t$$

(21) $$= \frac{1}{13}(3e^{3t} - 3\cos 2t + 2\sin 2t) \text{ Ans.}$$

CH-06 NUMERICAL METHODS

Definition :- A number 'ξ' is a solⁿ of $f(x)=0$, if $f(\xi)=0$ such a solⁿ 'ξ' is called a root or a zero of $f(x)=0$

→ Two types of Method

(i) Direct Method (ii) Iterative Method

Intermediate value Theorem :-

In this method, Let us consider any equation $y=f(x)$. Now we have to find out the value of the function $y=f(x)$, for a_0 & b_0.

Let $f(a_0)>0$ and $f(b_0)<0$

$\Rightarrow f(a_0)\cdot f(b_0)<0 \Rightarrow$ Root lies betⁿ a_0 & b_0.

Ex :- $y=f(x)=x^2-9$

$\Rightarrow f(0)=-9<0$ | $\Rightarrow f(3)=-1<0$

$\Rightarrow f(1)=-8<0$ | & $f(4)=7>0$

$\Rightarrow f(2)=-5<0$

Since $f(3)\cdot f(4)<0 \Rightarrow$ root lies betⁿ 3 & 4.

Bisection Method :-

This method is based on the repeated application of the intermediate value theorem.

→ Let $f(x) = 0$ be any eqn. locating betn 'a' and 'b'. If $f(x)$ is continuous betn 'a' & 'b' and $f(a)$ & $f(b)$ are of opposite sign, then there is a root betn 'a' & 'b'

→ Let $f(a)$ be -ve & $f(b)$ is +ve, then 1st approximation to the root is

$$x_1 = \frac{(a+b)}{2}$$

→ If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$, otherwise the root lies betn 'a' and 'x_1' or 'x_1' & 'b' according as $f(x_1)$ is +ve or -ve.

→ If $f(x_1)$ is +ve, so that the root lies betn a & x_1, then the 2nd approximation ~~lies betn~~ root is

$$x_2 = \frac{a + x_1}{2}$$

→ If $f(x_1)$ is -ve, root lies betn x_1 & x_2 then 3rd approximation $x_3 = \frac{x_1 + x_2}{2}$ and so on.

Ex:- Find the root of the eqn $x^4 - x - 10 = 0$

Soln:- Let $f(x) = x^4 - x - 10$

Here $f(0) = 0 - 0 - 10 = -10$ (-ve)

$f(1) = 1 - 1 - 10 = -10$ (-ve)

$f(2) = 2^4 - 2 - 10 = 16 - 12 = 4$ (+ve)

$\Rightarrow$ A root lies betn 1 & 2.

$\therefore$ 1st approximation is $x_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5$

Then $f(x_1) = f(1.5) = (1.5)^4 - 1.5 - 10 = 5.0625 - 11.5$

$\Rightarrow f(x_1) = -6.4375$ (-ve)

$\Rightarrow$ Root lies betn x_1 & 2.

$\therefore$ 2nd Approximation $x_2 = \frac{x_1 + 2}{2} = \frac{1.5+2}{2}$

$\Rightarrow x_2 = \frac{3.5}{2} = 1.75$

Then $f(x_2) = f(1.75) = (1.75)^4 - 1.75 - 10$

$\Rightarrow f(x_2) = -2.371$ (-ve)

$\Rightarrow$ Root lies betn 2 & x_2

$\therefore$ 3rd approximation is

$x_3 = \frac{x_2 + 2}{2} = \frac{1.75+2}{2} = 1.875$

Then $f(x_3) = f(1.875) = (1.875)^3 - 1.875 - 10$

$= 0.4846$ (+ve)

$\Rightarrow$ Root lies betn x_2 & x_3

Hence the 4th approximation is

$$x_4 = \frac{x_2 + x_3}{2} = \frac{1.75 + 1.875}{2} = 1.8125$$

Hence, the root is 1.81 correct to two decimal places approximately.

Newton - Raphson Method :-

Newton Raphson Method is based on the 1st deg. eqn and it's 1st derivative.

Suppose $f(x) = 0$ is a 1st deg. eqn in x.

then $f(x) = a_0 x + a_1 = 0 \Rightarrow a_0 x = -a_1$

$$\Rightarrow x = -\frac{a_1}{a_0} \quad \text{— (i)}$$

→ We find an approximate root of the eqn (i) if x_k is the approximation to the root then we define a_0 & a_1

→ Now we can find next approximation root, i.e x_{k+1}:

$$f(x_k) = f_k = a_0 x_k + a_1 \quad \text{— (ii)}$$

$$f'(x_k) = f'_k = a_0 \quad \text{— (iii)}$$

From (ii) we have, $a_1 = f_k - a_0 x_k$

$$\Rightarrow a_1 = f_k - f'_k x_k \quad \text{— (iv)}$$

putting (v) and (iii) in desired soln we get

$x_{k+1} = -\frac{a_1}{a_0}$; we have

$$x_{k+1} = \frac{f'_k x_k - f_k}{f'_k}$$

$$\boxed{\Rightarrow x_{k+1} = x_k - \frac{f_k}{f'_k}}$$ which is necessary formula for N-R method.

Ex:- Find by Newton's method, a root of the eqn $x^3-3x+1=0$ correct to 3 decimal places.

Soln Let $f(x) = x^3-3x+1$

$\Rightarrow f(1) = 1-3+1 = -1$ (-ve)

$f(2) = 2^3-3\cdot2+1 = 8-6+1 = 3$ (+ve)

$\Rightarrow$ The root of $f(x)$ lies betn 1 & 2

So 1st approximation $x_0 = \frac{1+2}{2} = 1.5$

Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \left(\frac{x_n^3-3x_n+1}{3x_n^2-3}\right)$$

$$= \frac{3x_n^3-3x_n-x_n^3+3x_n-1}{3x_n^2-3} = \frac{2x_n^3-1}{3x_n^2-3} \quad \text{—(1)}$$

Putting $n=0$ in (1) the 1st approximation

$$x_1 = \frac{2x_0^3-1}{3x_0^2-3} = \frac{2\times(1.5)^3-1}{3\times(1.5)^2-3} = \frac{5.75}{3.75} = 1.533$$

Putting $n=2$, the 2nd approximation x_2 is given by $x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 3}$

$$= \frac{2\times(1.533)^3 - 1}{3\times(1.533)^2 - 3} = \frac{6.2052}{4.05} = 1.532$$

Ans